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MECANICĂ

CAIET DE SEMINAR

VOLUMUL I

STATICA ŞI CINEMATICA

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CAPITOLUL 1

NOȚIUNI DE CALCUL VECTORIAL

1.1. Generalități. Definiții

Mecanica operează cu mărimi fizice scalare şi cu mărimi fizice vectoriale.

Mărimea fizică scalară sau *scalarul* este mărimea fizică, care este caracterizată prin *valoare numerică absolută* sau *modul*.

Mărimea fizică vectorială sau *vectorul* este mărimea fizică, care se caracterizează prin *valoare numerică absolută (modul)*, şi *orientare* adică *direcție şi sens*.

Mărimile fizice scalare (scalarii) şi *mărimile fizice vectoriale (vectorii)* la un loc se mai numesc şi *mărimi fizice tensoriale* sau *tensori*.

Aşadar vectorul este segmentul de dreaptă orientat (fig. 1.1), cu patru elemente caracteristice: origine sau punct de aplicație *A*, direcție sau dreaptă suport *Δ*, sens şi modul (mărime, intensitate, urmă) *v*.

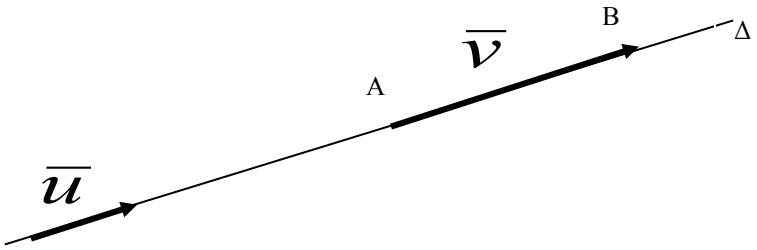

Fig. 1.1: Componența unui vector

Versorul este vectorul de modul unitar şi este dat de relația:

$$\bar{u} = \frac{\bar{v}}{v} = \frac{\bar{v}}{|\bar{v}|}$$

Definim componentele pe axele *Ox*, *Oy* şi *Oz* ale versorului din relația precedentă astfel:

$$\begin{cases} \bar{i} = pr_{Ox}\bar{u} \\ \bar{j} = pr_{Oy}\bar{u} \\ \bar{k} = pr_{Oz}\bar{u} \end{cases}$$

Un vector oarecare se scrie în funcție de componentele pe axe ale versorului său astfel:

$$\bar{v} = v_x\bar{i} + v_y\bar{j} + v_z\bar{k}$$

unde

$$\begin{cases} v_x = pr_{Ox}\bar{v} \\ v_y = pr_{Oy}\bar{v} \\ v_z = pr_{Oz}\bar{v} \end{cases}$$

sau:

$$\bar{v} = (v_x, v_y, v_z)$$

1.2. Proiecția unui vector pe o axă

Proiecția unui vector pe o axă se defineşte ca fiind produsul dintre vectorul respectiv şi cosinusul unghiului pe care direcția acestuia îl face cu axa pe care se proiectează vectorul studiat (fig. 1.2).

$$pr_{\Delta}\bar{v} = v\cos\alpha$$

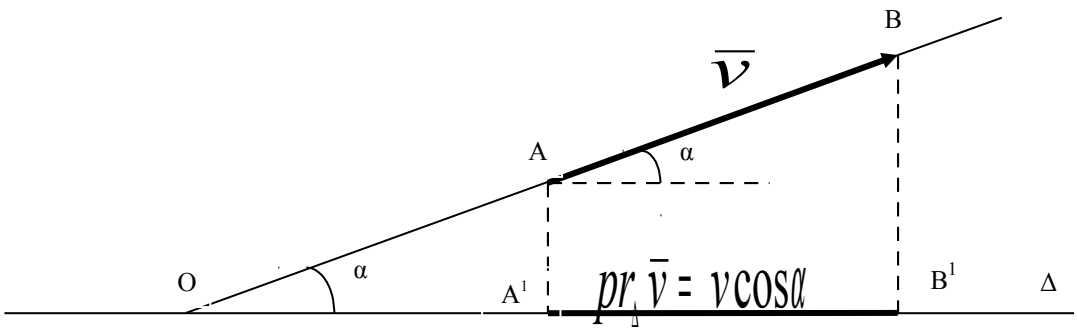

Fig. 1.2: Proiecția unui vector pe o axă

1.3. Operații cu vectori

1.3.1. Adunarea (compunerea) vectorilor

Fie vectorii:

$\bar{a} = \bar{a}(a_x, a_y, a_z)$; $\quad \bar{b} = \bar{b}(b_x, b_y, b_z)$

atunci se defineşte suma celor doi vectori:

$$\bar{c} = \bar{a} + \bar{b}; \qquad \bar{c} = \bar{c}(c_x, c_y, c_z) = \bar{c}[(a_x + b_x), (a_y + b_y), (a_z + b_z)]$$

Ca mărime, suma celor doi vectori se scrie astfel:

$$c = \sqrt{a^2 + b^2 + 2ab\cos\alpha}$$

unde α este unghiul dintre cei doi vectori.

Adunarea mai multor vectori se realizează astfel:

$$\bar{c} = \bar{a}_1 + \bar{a}_2 + ... + \bar{a}_n$$

1.3.1.1. Interpretare geometrică

Geometric, adunarea a doi vectori se realizează cu regula paralelogramului sau cu regula triunghiului (fig. 1.3).

Compunerea mai multor vectori se realizează, din punct de vedere geometric, cu regula poligonului (fig. 1.4).

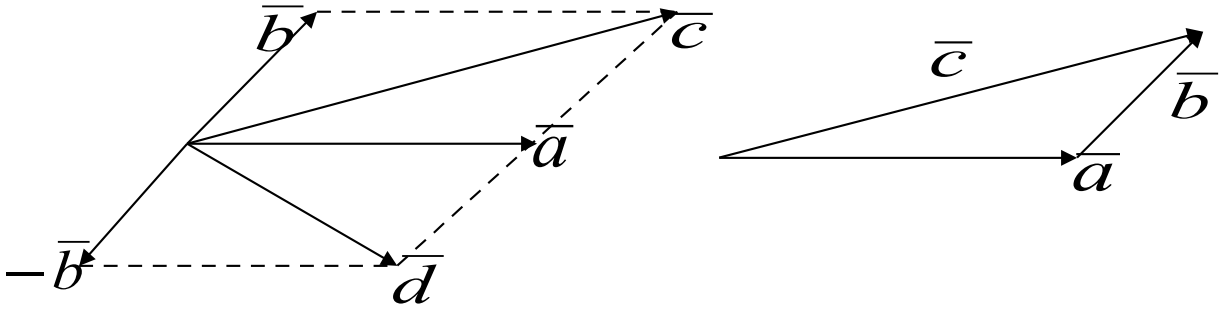

Fig. 1.3: Compunerea geometrică a doi vectori

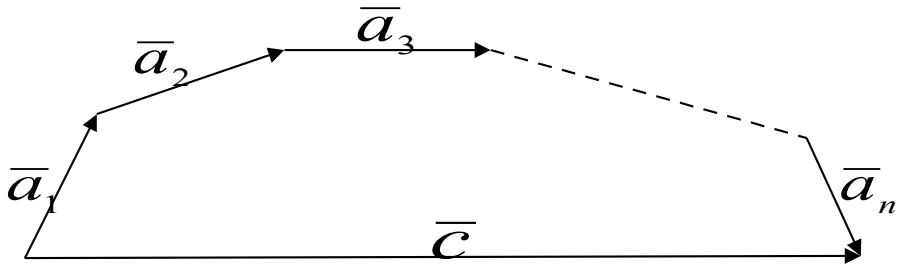

Fig. 1.4: Compunerea geometrică a mai multor vectori

1.3.1.2. Proprietăți

a) comutativitate;
b) are element neutru (vectorul nul).

1.3.2. Scăderea (diferența) vectorilor

Fie vectorii:

$$\bar{a} = \bar{a}(a_x, a_y, a_z); \qquad \bar{b} = \bar{b}(b_x, b_y, b_z)$$

atunci se definește diferența celor doi vectori (fig. 1.3):

$$\bar{d} = \bar{a} - \bar{b}; \qquad \bar{d} = \bar{d}(d_x, d_y, d_z) = \bar{d}[(a_x - b_x), (a_y - b_y), (a_z - b_z)]$$

1.3.2.1. Interpretare geometrică

$$\bar{d} = \bar{a} - \bar{b} = \bar{a} + \left(-\bar{b}\right)$$

1.3.2.2. Proprietăți

a) Element neutru: vectorul nul

1.3.3. Descompunerea unui vector după două direcții

Conform figurii 1.5. avem:

$$\bar{v} = \bar{v}_1 + \bar{v}_2 = v_1\bar{u}_1 + v_2\bar{u}_2$$

Pentru direcții ortogonale relația de mai sus devine (fig. 1.5.):

$$\bar{v} = \bar{v}_x + \bar{v}_y = v_x\bar{i} + v_y\bar{j}; \qquad v = \sqrt{v_1^2 + v_2^2}$$

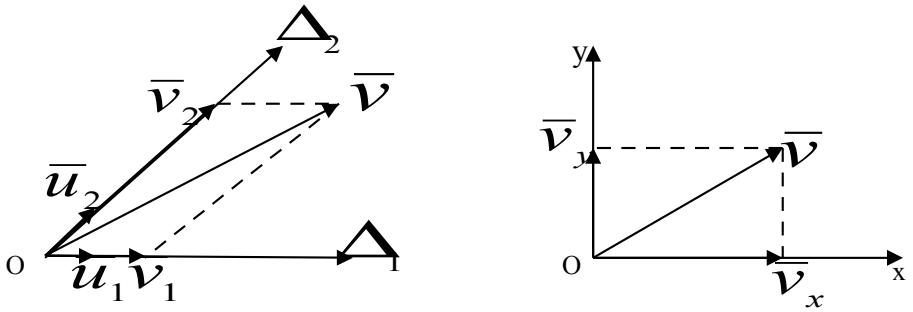

Fig. 1.5: Descompunerea unui vector după două direcții

1.3.4. Descompunerea unui vector după trei direcții

Din figura 1.6 avem:

$$\bar{v} = v_x\bar{i} + v_y\bar{j} + v_z\bar{k}; \qquad v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

unde v_x, v_y,v_z sunt proiecțiile vectorului studiat pe cele trei axe.

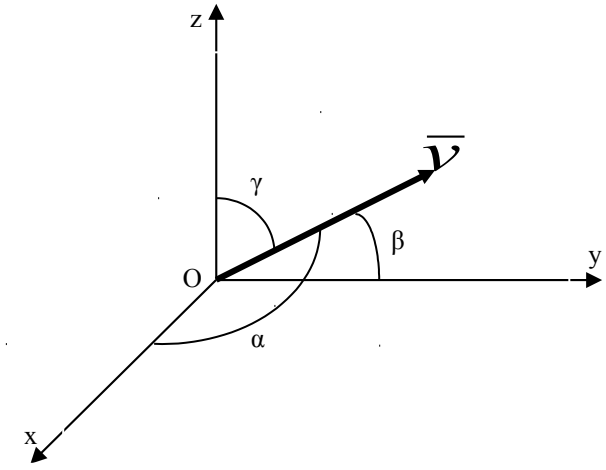

Fig. 1.6: Descompunerea unui vector după trei direcții ortogonale

$$\cos\alpha = \frac{v_x}{v}; \qquad \cos\beta = \frac{v_y}{v}; \qquad \cos\gamma = \frac{v_z}{v}$$

unde: *α, β, γ* sunt unghiurile directoare ale vectorului studiat față de axele *Ox, Oy, Oz*;
cosα, cosβ, cosγ sunt cosinuşii directori ai vectorului studiat față de axele *Ox, Oy, Oz*.

Din relația precedentă rezultă că:

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

1.3.5. Produsul scalar a doi vectori

Este mărimea scalară care reprezintă produsul modulelor celor doi vectori prin cosinusul unghiului dintre ei.

$$p=\bar{a}\bar{b}=ab\cos\alpha$$

unde α este unghiul dintre cei doi vectori.

Relația precedentă este echivalentă cu relația:

$$\cos\alpha=\frac{\bar{a}\bar{b}}{ab}$$

Dacă:

$$\bar{a}=a_x\bar{i}+a_y\bar{j}+a_z\bar{k};\qquad \bar{b}=b_x\bar{i}+b_y\bar{j}+b_z\bar{k}$$

Atunci:

$$p=\bar{a}\bar{b}=a_xb_x+a_yb_y+a_zb_z$$

De asemenea se observă că:

$$\bar{i}\cdot\bar{i}=\bar{j}\cdot\bar{j}=\bar{k}\cdot\bar{k}=1\qquad \bar{i}\cdot\bar{j}=\bar{j}\cdot\bar{k}=\bar{k}\cdot\bar{i}=0$$

1.3.5.1. Proprietăți

a) Comutativitate;
b) Distributivitate față de adunare.

1.3.5.2. Interpretare geometrică

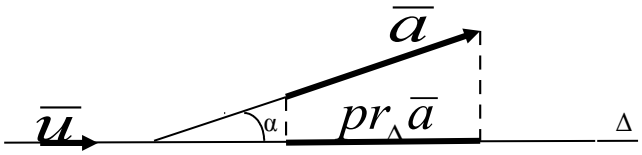

Fig. 1.7: Interpretarea geometrică a produsului scalar a doi vectori

$$\begin{aligned} a_x &= a\bar{i}\\ a_y &= a\bar{j}\\ pr_\Delta\bar{a} = a\cdot\cos\alpha &= \bar{a}\bar{u}\\ a_z &= a\bar{k}\end{aligned}$$

1.3.6. Produsul vectorial a doi vectori

Este un vector perpendicular pe planul celor doi vectori si definit ca fiind de mărime egală cu produsul mărimilor celor doi vectori prin sinusul unghiului dintre ei.

$$\bar{v} = \bar{a} \times \bar{b} \qquad v = ab \cdot \sin\alpha$$

$$\bar{v} \perp \bar{a}; \quad \bar{v} \perp \bar{b} \quad \Rightarrow \bar{v} \perp (\bar{a}; \bar{b})$$

1.3.6.1. Proprietăți

a) Anticomutativitate:

$$\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$$

b) Distributivitate față de adunare

c) Dacă:

$$\bar{a} \times \bar{b} = 0$$

Atunci cei doi vectori sunt paraleli sau coliniari.

1.3.6.2. Interpretare geometrică

Produsul vectorial a doi vectori este egal cu aria paralelogramului determinat de cei doi vectori.

Demonstrație (vezi figura 1.8)

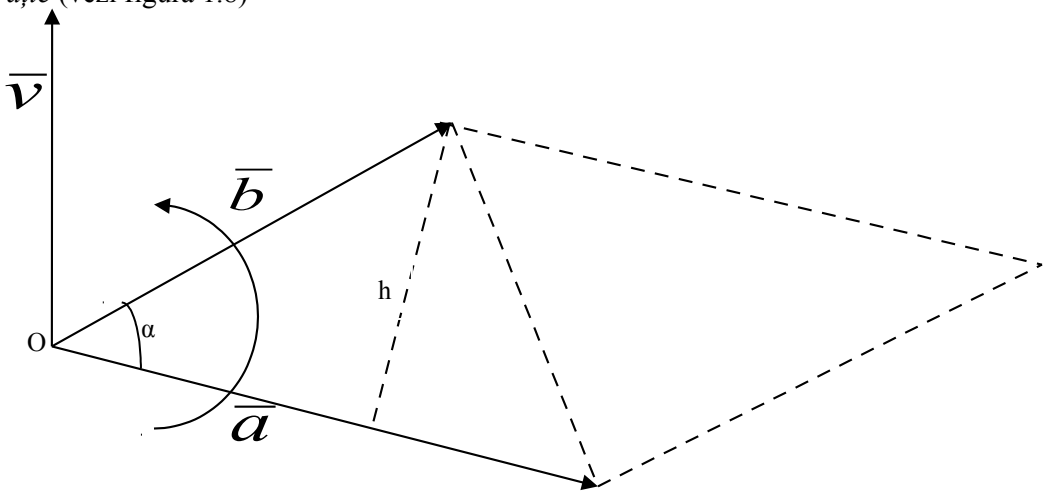

Fig. 1.8: Interpretarea geometrică a produsului vectorial a doi vectori

$$A_{tr} = \frac{1}{2} \cdot ah = \frac{1}{2} \cdot ab \cdot \sin\alpha \Rightarrow A_{par} = 2A_{tr} = ab \cdot \sin\alpha = \bar{a} \times \bar{b}$$

unde: A_{tr} – aria triunghiului determinat de cei doi vectori;
A_{par} – aria paralelogramului determinat de cei doi vectori.

1.3.6.3. Determinare analitică

$$\bar{v} = \bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

1.3.6.4. Observații

$$\bar{a} \times \bar{a} = \bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0 \qquad\qquad \bar{i} \times \bar{j} = \bar{k}$$

1.3.7. Produsul mixt a trei vectori

Reprezintă mărimea fizică scalară egală cu produsul scalar dintre un vector şi produsul vectorial al celorlalți doi.

$$w = \bar{a}(\bar{b} \times \bar{c}) = \bar{a}, \bar{b}, \bar{c} = (\bar{a}, \bar{b}, \bar{c}) = \bar{c}(\bar{a} \times \bar{b}) = \bar{b}(\bar{c} \times \bar{a})$$

1.3.7.1. Interpretare geometrică

Produsul mixt a trei vectori reprezintă volumul paralelipipedului determinat de cei trei vectori.

Demonstrație (vezi figura 1.9)

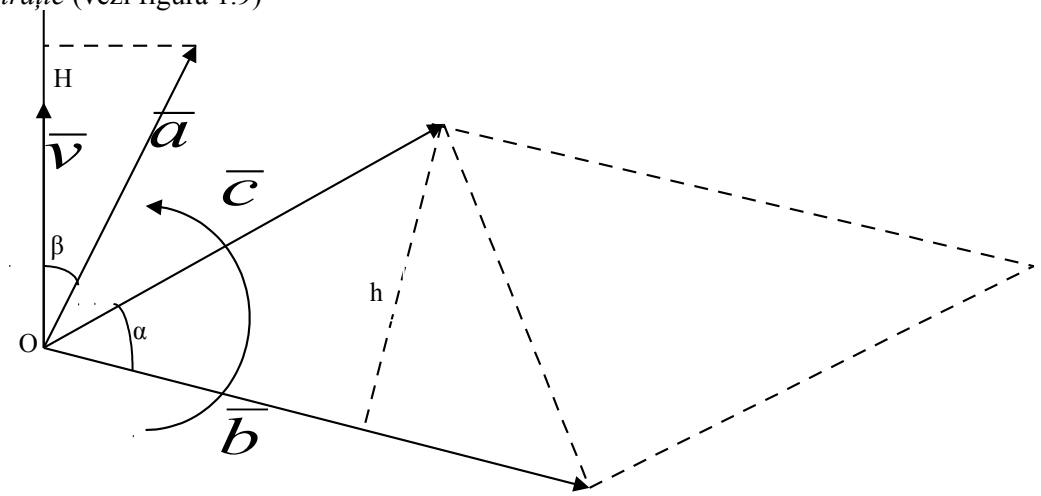

Fig. 1.9: Interpretarea geometrică a produsului mixt a trei vectori

$$w = \bar{a}(\bar{b} \times \bar{c}) = \bar{a} \cdot \bar{v} = a \cdot v \cdot \cos\beta = v \cdot (a \cdot \cos\beta) = A_{par} \cdot H = V_{paralelipiped}$$

1.3.7.2. Determinare analitică

$$w = \bar{a}(\bar{b} \times \bar{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

1.3.8. Dublul produs vectorial

Reprezintă mărimea fizică vectorială egală cu produsul vectorial dintre un vector şi produsul vectorial al celorlalți doi.

$$\bar{q} = \bar{a} \times (\bar{b} \times \bar{c}) = \bar{a} \times \bar{v}$$

1.4. Probleme rezolvate:

1) Fie vectorii:

$\bar{a} = 2\bar{i} - 3\bar{j} + 2\bar{k}; \qquad \bar{b} = \bar{i} + 4\bar{j} - \bar{k}; \qquad \bar{c} = 3\bar{i} + 2\bar{j}$

Se cer:

Rezolvare:

$\bar{a} + \bar{b} = 3\bar{i} + \bar{j} + \bar{k}; \bar{a} - \bar{b} = \bar{i} - 7\bar{j} + 3\bar{k}; \bar{a} + \bar{c} = 5\bar{i} - \bar{j} + 2\bar{k};$

$\bar{a} - \bar{c} = -\bar{i} - 5\bar{j} + 2\bar{k}; \bar{a} + \bar{b} + \bar{c} = 6\bar{i} + 3\bar{j} + \bar{k}; \bar{a} + \bar{b} - \bar{c} = -\bar{j} + \bar{k};$

$\bar{a}\bar{b} = 2 \cdot 1 - 3 \cdot 4 + 2 \cdot (-1) = -12; \bar{b}\bar{c} = 1 \cdot 3 + 4 \cdot 2 - 1 \cdot 0 = 11;$

$\bar{b} \cdot \bar{a} = 1 \cdot 2 + 4 \cdot (-3) - 1 \cdot 2 = -12; \bar{c}\bar{b} = 3 \cdot 1 + 2 \cdot 4 + 0 \cdot (-1) = 11;$

$\bar{u}_a = \frac{\bar{a}}{a} = \frac{2}{\sqrt{17}}\bar{i} - \frac{3}{\sqrt{17}}\bar{j} + \frac{2}{\sqrt{17}}\bar{k}; \bar{u}_b = \frac{\bar{b}}{b} = \frac{1}{\sqrt{18}}\bar{i} + \frac{4}{\sqrt{18}}\bar{j} - \frac{1}{\sqrt{18}}\bar{k}; \bar{u}_c = \frac{\bar{c}}{c} = \frac{3}{\sqrt{13}}\bar{i} + \frac{2}{\sqrt{13}}\bar{j}.$

2)

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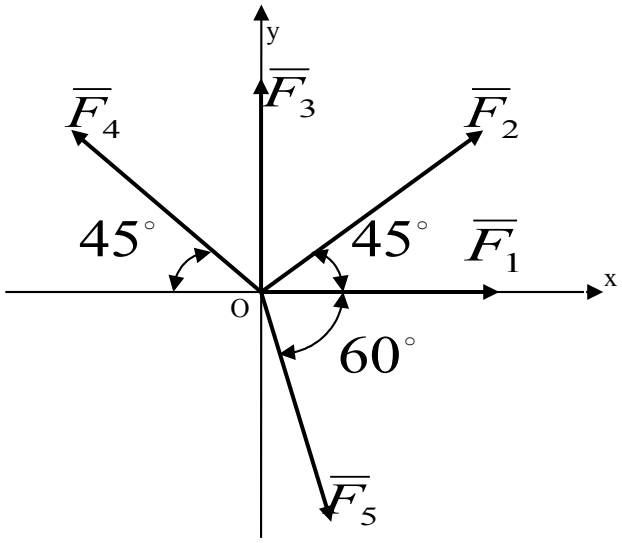

Fig. 1.10

Se cer:

Rezolvare:

$\bar{R} = \sum_{i=1}^{5} \bar{F}_i = R_x\bar{i} + R_y\bar{j} = \left(\sum_{i=1}^{5} F_{ix}\right) \cdot \bar{i} + \left(\sum_{i=1}^{5} F_{iy}\right) \cdot \bar{j};$

$R_x = \sum_{i=1}^{5} F_{ix} = F_{1x} + F_{2x} + F_{3x} + F_{4x} + F_{5x} = P + 2P\frac{\sqrt{2}}{2} - 2P\frac{\sqrt{2}}{2} + 2P\frac{1}{2} = 2P;$

$R_y = \sum_{i=1}^{5} F_{iy} = F_{1y} + F_{2y} + F_{3y} + F_{4y} + F_{5y} = 2P\frac{\sqrt{2}}{2} + P\sqrt{3} + 2P\frac{\sqrt{2}}{2} - 2P\frac{\sqrt{3}}{2} = 2P\sqrt{2};$

$\bar{R} = R_x\bar{i} + R_y\bar{j} = 2P\bar{i} + 2P\sqrt{2}\bar{j}; R = \sqrt{R_x^2 + R_y^2} = \sqrt{4P^2 + 8P^2} = \sqrt{12P^2} = 2P\sqrt{3};$

$tg\alpha = \frac{R_y}{R_x} = \sqrt{2}; \bar{u}_R = \frac{\bar{R}}{R} = \frac{2P\bar{i} + 2P\sqrt{2}\bar{j}}{2P\sqrt{3}} = \frac{1}{\sqrt{3}}\bar{i} + \frac{\sqrt{2}}{\sqrt{3}}\bar{j}.$

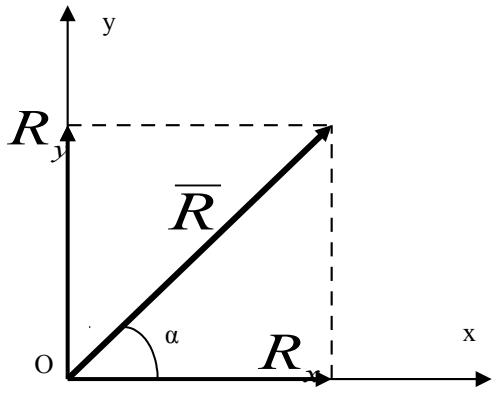

Fig. 1.11.

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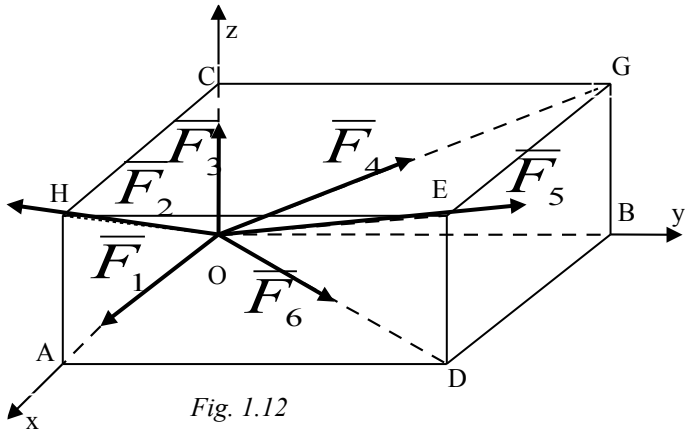

Fig. 1.12

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OA=2l; OB=3l; OC=l;

Se cer:

Rezolvare:

$O(0,0,0); H(2l,0,l); G(0,3l,l); E(2l,3l,l); D(2l,3l,0);$

$$\overline{F}_2 = F_2\overline{u}_2 = F_2\frac{\overline{OH}}{OH} = F_2\frac{(x_H - x_O)\cdot\bar{i} + (y_H - y_O)\cdot\bar{j} + (z_H - z_O)\cdot\bar{k}}{\sqrt{(x_H - x_O)^2 + (y_H - y_O)^2 + (z_H - z_O)^2}} = 2P\sqrt{5}\frac{2l\bar{i} + l\bar{k}}{l\sqrt{4+1}} = 4P\bar{i} + 2P\bar{k};$$

$$\overline{F}_4 = F_4\overline{u}_4 = F_4\frac{\overline{OG}}{OG} = F_4\frac{(x_G - x_O)\cdot\bar{i} + (y_G - y_O)\cdot\bar{j} + (z_G - z_O)\cdot\bar{k}}{\sqrt{(x_G - x_O)^2 + (y_G - y_O)^2 + (z_G - z_O)^2}} = P\sqrt{10}\frac{3l\bar{j} + l\bar{k}}{l\sqrt{9+1}} = 3P\bar{j} + P\bar{k};$$

$$\overline{F}_5 = F_5\overline{u}_5 = F_5\frac{\overline{OE}}{OE} = F_5\frac{(x_E - x_O)\cdot\bar{i} + (y_E - y_O)\cdot\bar{j} + (z_E - z_O)\cdot\bar{k}}{\sqrt{(x_E - x_O)^2 + (y_E - y_O)^2 + (z_E - z_O)^2}} = P\sqrt{14}\frac{2l\bar{i} + 3l\bar{j} + l\bar{k}}{l\sqrt{4+9+1}} = 2P\bar{i} + 3P\bar{j} + P\bar{k};$$

$$\overline{F}_6 = F_6\overline{u}_6 = F_6\frac{\overline{OD}}{OD} = F_6\frac{(x_D - x_O)\cdot\bar{i} + (y_D - y_O)\cdot\bar{j} + (z_D - z_O)\cdot\bar{k}}{\sqrt{(x_D - x_O)^2 + (y_D - y_O)^2 + (z_D - z_O)^2}} = 2P\sqrt{13}\frac{2l\bar{i} + 3l\bar{j}}{l\sqrt{4+9}} = 4P\bar{i} + 6P\bar{j}.$$

Rezultatele obținute se centralizează în următorul tabel:

Forțe	F_{ix}	F_{iy}	F_{iz}
F_1	P	0	0
F_2	4P	0	2P
F_3	0	0	2P
F_4	0	3P	P
F_5	2P	3P	P
F_6	4P	6P	0
$\sum$	11P	12P	6P

Rezultă:

$$\bar{R} = 11P\bar{i} + 12P\bar{j} + 6P\bar{k}; R = \sqrt{R_x^2 + R_y^2 + R_z^2} = P\sqrt{11^2 + 12^2 + 6^2} = P\sqrt{121+144+36} = P\sqrt{301};$$

$$\cos\alpha = \frac{R_x}{R} = \frac{11P}{P\sqrt{301}} = \frac{11}{\sqrt{301}}; \cos\beta = \frac{R_y}{R} = \frac{12P}{P\sqrt{301}} = \frac{12}{\sqrt{301}}; \cos\gamma = \frac{R_z}{R} = \frac{6P}{P\sqrt{301}} = \frac{6}{\sqrt{301}};$$

$$\bar{u}_R = \frac{\bar{R}}{R} = \frac{11P\bar{i} + 12P\bar{j} + 6P\bar{k}}{P\sqrt{301}} = \frac{11}{\sqrt{301}}\bar{i} + \frac{12}{\sqrt{301}}\bar{j} + \frac{6}{\sqrt{301}}\bar{k}.$$

1.5. Probleme propuse:

1) Fie vectorii:

$\bar{a} = \bar{a}(1,-2,3); \bar{b} = \bar{b}(-2,4-6).$

Se cer:

$\bar{a} + \bar{b} = ?; \bar{a} - \bar{b} = ?; \bar{a} \cdot \bar{b} = ?; \bar{a} \times \bar{b} = ?; \bar{u}_a = ?; \bar{u}_b = ?.$

2)

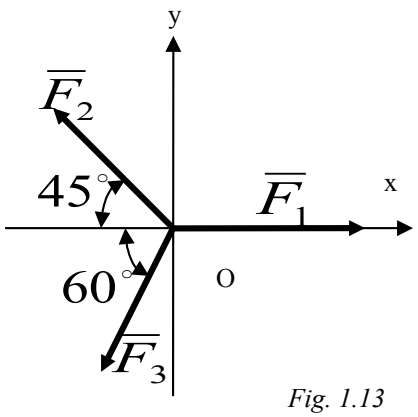

Fig. 1.13

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Se cer:

Răspuns:

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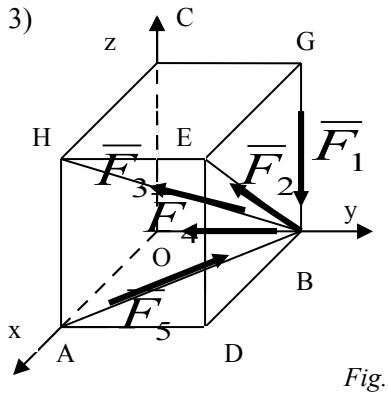

Fig. 1.14

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Se cer:

Răspuns:

$$\overline{R} = 6P\bar{i} - 7P\bar{j} + 14P\bar{k}; R = P\sqrt{281}; \overline{u}_R = \frac{6}{\sqrt{281}}\bar{i} - \frac{7}{\sqrt{281}}\bar{j} + \frac{14}{\sqrt{281}}\bar{k};$$

$$\cos\alpha = \frac{6}{\sqrt{281}}; \cos\beta = \frac{-7}{\sqrt{281}}; \cos\gamma = \frac{14}{\sqrt{281}}.$$

CAPITOLUL 2

STATICA PUNCTULUI MATERIAL

2.1. Generalități. Definiții

Statica este capitolul mecanicii care studiază echilibrul corpurilor sub acțiunea sistemelor de forțe. Un corp este în echilibru dacă şi numai dacă rezultanta forțelor care acționează asupra sa este nulă, adică:

$$\overline{R} = \sum_{i=1}^{n} \overline{F}_i = 0$$

Descompunându-se pe axele de coordonate se obține:

$$\begin{cases} R_x = \sum_{i=1}^{n} F_{ix} = 0 \\ R_y = \sum_{i=1}^{n} F_{iy} = 0 \\ R_z = \sum_{i=1}^{n} F_{iz} = 0 \end{cases}$$

$$F_f \leq \mu \cdot N$$

Relațiile precedente sunt valabile numai în cazul în care punctul material este acționat de orice alte forțe în afară de forțele de frecare. În absența forțelor de frecare avem legea lui Coulomb:

2.2. Statica punctului material liber

2.2.1. Probleme rezolvate

1)

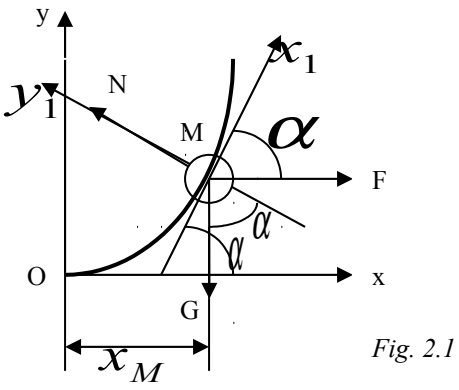


Fig. 2.1

Date:

Se cer:
Poziția de echilibru
Reacțiunea N=?

Rezolvare:

$$\overline{F} + \overline{G} + \overline{N} = 0$$

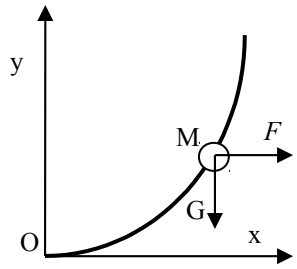
$$(Mx_1): F\cos\alpha - G\sin\alpha = 0$$
$$(My_1): N - F\sin\alpha - G\cos\alpha = 0$$

$$tg\alpha = y' = \frac{dy}{dx} = \frac{F}{G} = \frac{2 \cdot x_M}{c} \Rightarrow x_M = \frac{cF}{2G}$$

$$N = F\sin\alpha + G\cos\alpha = F\frac{tg\alpha}{\sqrt{1+tg^2\alpha}} + G\frac{1}{\sqrt{1+tg^2\alpha}} = F\frac{F/G}{\sqrt{1+\left(F/G\right)^2}} + G\frac{1}{\sqrt{1+\left(F/G\right)^2}}$$

2.2.2. Probleme propuse

1)



Date:
M(x,y);

Se cer:

Fig. 2.2

Răspuns:

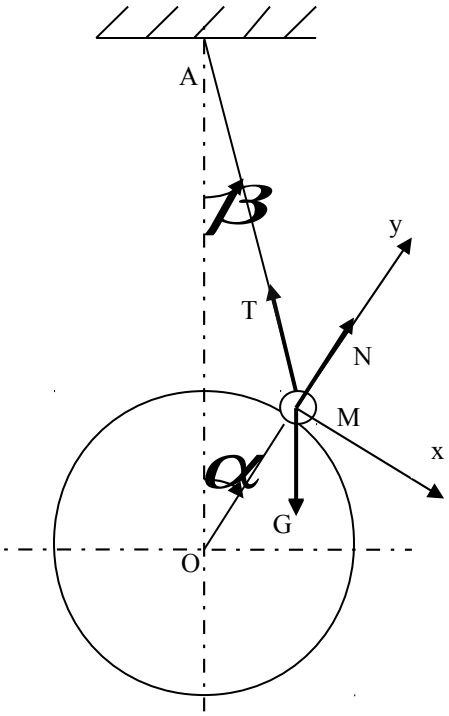
$x = l; y = 2l; N = G\sqrt{17}.$

2.3. Statica punctului material supus la legături

2.3.1. Statica punctului material supus la legături fără frecare

2.3.1.1. Probleme rezolvate

1)



Date:

Se cer:

Fig. 2.3

Rezolvare:

$$\overline{G} + \overline{T} + \overline{N} = 0$$

$$\sum F_{ix} = 0 : G\sin\alpha - T\sin(\alpha+\beta) = 0$$
$$\sum F_{iy} = 0 : N + T\cos(\alpha+\beta) - G\cos\alpha = 0$$

$$T = \frac{G\sin\alpha}{\sin(\alpha+\beta)}; N = \frac{G\sin\beta}{\sin(\alpha+\beta)}$$

2)

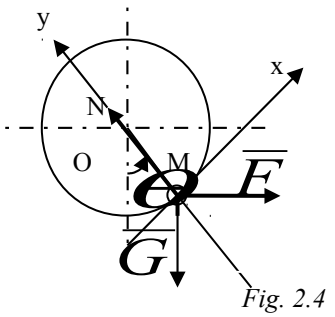


Fig. 2.4

Date:

Se cer:

Rezolvare:

$$\overline{F} + \overline{G} + \overline{N} = 0$$

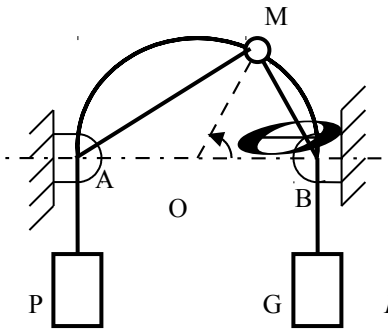
$$\sum F_{ix} = 0 : F\cos\theta - G\sin\theta = 0$$
$$\sum F_{iy} = 0 : N - F\sin\theta - G\cos\theta = 0$$

$$tg\theta = \frac{F}{G}; N = \sqrt{F^2 + G^2}.$$

2.3.1.2. Probleme propuse

1)

Date:



Se cer:
pentru echilibru.

Răspuns:

Fig. 2.5

2.3.2. Statica punctului material supus la legături cu frecare

2.3.2.1. Probleme rezolvate

1)

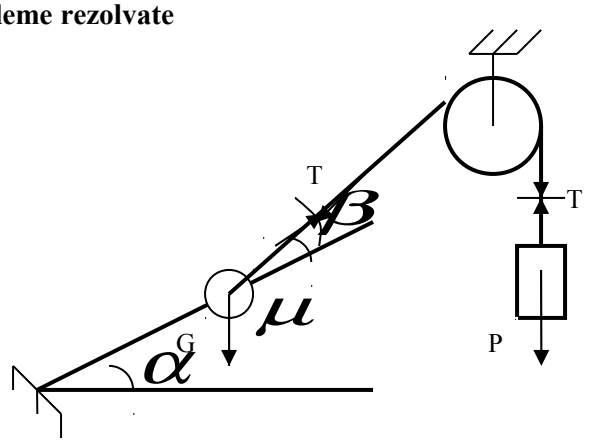


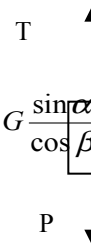
Fig. 2.6

Date:

Se cer:
N=?;P=?
pentru
echilibru

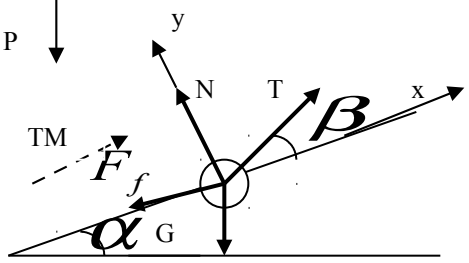
Rezolvare:

$$P \geq G\frac{\sin\alpha - \mu\cdot\cos\alpha}{\cos\beta - \mu\cdot\sin\beta}$$



$$G\frac{\sin\alpha - \mu\cdot\cos\alpha}{\cos\beta - \mu\cdot\sin\beta} \leq P \Leftrightarrow \sum F_{iy} = 0 \Rightarrow T\sin\alpha + P\mu - \cos\alpha \Rightarrow T = P\frac{\cos\alpha + \mu\cdot\sin\alpha}{\cos\beta + \mu\cdot\sin\beta}$$

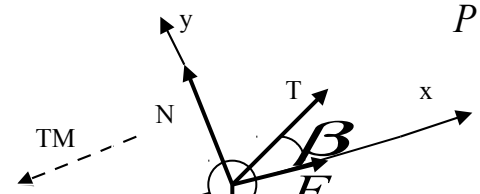
2)



$$\sum F_{ix} = 0 : T\cos\beta - F_f - G\sin\alpha = 0$$
$$\sum F_{iy} = 0 : N - G\cos\alpha + T\sin\beta = 0$$
$$F_f \leq \mu\cdot N$$

$$F_f = P\cos\beta - G\sin\alpha \qquad P\cos\beta - G\sin\alpha \leq \mu\cdot(G\cos\alpha - P\sin\beta)$$
$$N = G\cos\alpha - P\sin\beta \qquad P\cdot(\cos\beta + \mu\sin\beta) \leq G\cdot(\sin\alpha + \mu\cdot\cos\alpha)$$

$$P \leq G\frac{\sin\alpha + \mu\cdot\cos\alpha}{\cos\beta + \mu\cdot\sin\beta}$$

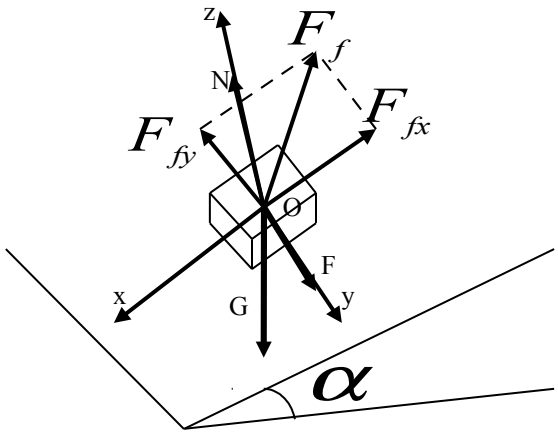


$$\sum F_{ix} = 0 : T\cos\beta + F_f - G\sin\alpha = 0$$
$$\sum F_{iy} = 0 : N + T\sin\beta - G\cos\alpha = 0$$

$F_f \le \mu \cdot N$

$F_f = G\sin\alpha - P\cos\beta \qquad G\sin\alpha - P\cos\beta \le \mu \cdot (G\cos\alpha - P\sin\beta)$

$N = G\cos\alpha - P\sin\beta \qquad P \cdot (\cos\beta - \mu \cdot \sin\beta) \ge G \cdot (\sin\alpha - \mu \cdot \cos\alpha)$

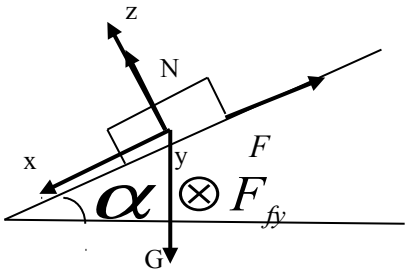


Date:

Se cere:
pentru echilibru

Fig. 2.7

Rezolvare:



$\sum F_{ix} = 0 : G\sin\alpha - F_{fx} = 0$

$\sum F_{iy} = 0 : F - F_{fy} = 0$

$\sum F_{iz} = 0 : N - G\cos\alpha = 0$

$F_{fx} = G\sin\alpha$

$F_{fy} = F$

$N = G\cos\alpha$

$F_f = \sqrt{F_{fx}^2 + F_{fy}^2} \le \mu \cdot N \Rightarrow \sqrt{G^2\sin^2\alpha + F^2} \le \mu \cdot G \cdot \cos\alpha \Rightarrow$

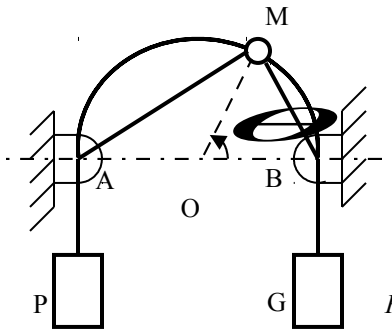
$\Rightarrow F^2 \le G^2(\mu^2\cos^2\alpha - \sin^2\alpha) \Rightarrow F \le G\sqrt{\mu^2\cos^2\alpha - \sin^2\alpha}$

Discuție:

2.3.2.2. Probleme propuse

1)

Date:

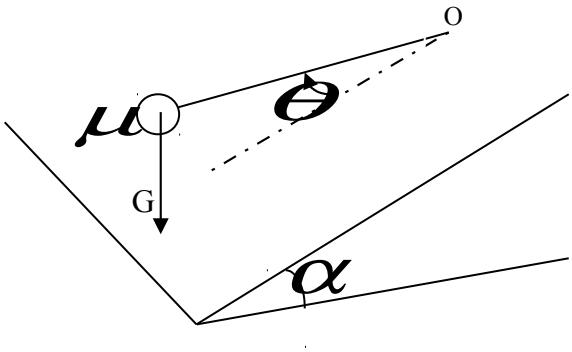


Se cere:
pentru echilibru.

Răspuns:

Fig. 2.8

2)



Date:

Se cere:
pentru echilibru.

Fig. 2.9

Răspuns:

$$\sin\theta \le \frac{\mu}{tg\alpha}; \sin\theta \le 1 \Rightarrow tg\varphi \le tg\alpha \Rightarrow \varphi \le \alpha$$

$$\varphi \ge \alpha \Rightarrow echilibru$$

CAPITOLUL 3

REDUCEREA SISTEMELOR DE FORȚE OARECARE APLICATE SOLIDULUI RIGID

3.1. Generalități. Definiții

3.1.1. Cazuri de reducere

1) $\overline{R} = 0; \overline{M}_O = 0 \Rightarrow$ echilibru;

2) $\overline{R} = 0; \overline{M}_O \neq 0 \Rightarrow$ cuplu;

3) $\overline{R} \neq 0; \overline{M}_O = 0 \Rightarrow$ rezultantă unică pe axa centrală;

4) $\overline{R} \neq 0; \overline{M}_O \neq 0 \Rightarrow$ a) $\overline{R} \cdot \overline{M}_O = 0 \Rightarrow$ rezultantă unică pe axa centrală;

b) $\overline{R} \cdot \overline{M}_O \neq 0 \Rightarrow$ torsor minimal.

3.1.2. Axa centrală

Reprezintă locul geometric al punctelor, în care se obține torsorul minimal după reducere şi are ecuațiile:

$$\frac{M_{Ox}-yR_z+zR_y}{R_x}=\frac{M_{Oy}-zR_x+xR_z}{R_y}=\frac{M_{Oz}-xR_y+yR_x}{R_z};M_{Oz}-xR_y+yR_x=0$$

3.2. Reducerea sistemelor de forțe spațiale

3.2.1. Probleme rezolvate

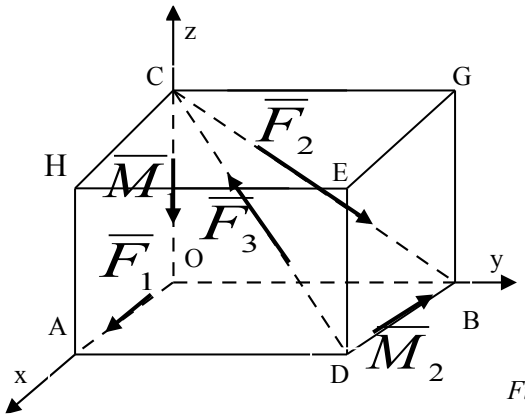


Fig. 3.1

Date:

Se cer:
echivalența=?; axa centrală?

Rezolvare:

$A(5l,0,0);\ B(0,3l,0);\ C(0,0,4l);\ D(5l,3l,0);$

$\overline{F}_i, \overline{M}_i$	F_{ix}	F_{iy}	F_{iz}	M_x	M_y	M_z
F_1	10P	0	0	0	0	0
F_2	0	6P	- 8P	-24Pl	0	0
F_3	- 10P	- 6P	8P	24Pl	- 40Pl	0
M_1	–	–	–	0	0	- Pl
M_2	–	–	–	- 2Pl	0	0
Σ	0	0	0	- 2Pl	- 40 Pl	- Pl

$$\overline{F}_3 = F_3 \cdot \frac{\overline{DC}}{DC} = 10\sqrt{2}P\frac{\left(-5l\bar{i}-3l\bar{j}+4l\bar{k}\right)}{\sqrt{25l^2+9l^2+16l^2}} = -10P\bar{i}+6P\bar{j}+8P\bar{k}$$

$$\overline{F}_2 = F_2 \cdot \frac{\overline{CB}}{CB} = 10P\frac{3l\bar{j}-4l\bar{k}}{\sqrt{9l^2+16l^2}} = 6P\bar{j}-8P\bar{k}$$

$$\overline{M}_O(\overline{F}_2) = \overline{OC} \times \overline{F}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & 4l \\ 0 & 6P & -8P \end{vmatrix} = -24Pl\bar{j};$$

$$\overline{M}_O(\overline{F}_3) = \overline{OD} \times \overline{F}_3 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5l & 3l & 0 \\ -10P & -6P & 8P \end{vmatrix} = 24Pl\bar{j} + 24Pl\bar{i}.$$

Axa centrală nu are sens fizic deoarece rezultanta sistemului de forțe este nulă.
Cazul de reducere: Cuplu.

2)

$$\begin{cases} \overline{R} = 0 \\ \overline{M}_O = 2Pl\bar{i} - 40Pl\bar{j} - Pl\bar{k} \end{cases}$$

$$\begin{cases} \overline{M}_E = \overline{M}_O - \overline{OE} \times \overline{R} = 2Pl\bar{i} - 40Pl\bar{j} - Pl\bar{k} - \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5l & 3l & 4l \\ 0 & 0 & 0 \end{vmatrix} = 2Pl\bar{i} - 40Pl\bar{j} - Pl\bar{k} \end{cases}$$

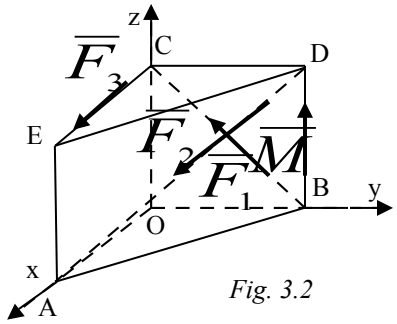


Fig. 3.2

Date:

Se cer:
echivalența=?; axa centrală=?.

Rezolvare:

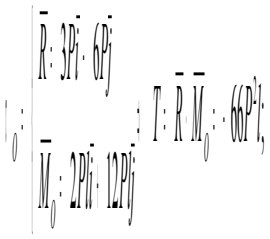
$$\overline{F}_2 = F_2\overline{u}_{BC} = F_2\frac{\overline{BC}}{BC} = F_2\frac{(x_C-x_B)\bar{i}+(y_C-y_B)\bar{j}+(z_C-z_B)\bar{k}}{\sqrt{(x_C-x_B)^2+(y_C-y_B)^2+(z_C-z_B)^2}} = -3P\bar{j}+4P\bar{k}$$

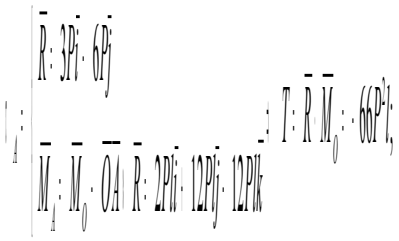
$$M_O(F_2) = \overline{OB} \times \overline{F}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_B & y_B & z_B \\ F_{2x} & F_{2y} & F_{2z} \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2l & 0 & 0 \\ 2P & -3P & -4P \end{vmatrix} = -8Pl\bar{j} - 6Pl\bar{k};$$

$$M_O(F_3) = \overline{OC} \times \overline{F}_3 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_C & y_C & z_C \\ F_{3x} & F_{3y} & F_{3z} \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & 4l \\ P & 0 & 0 \end{vmatrix} = 4Pl\bar{j};$$

$$M_O(F_3) = F_3 \cdot OC = P \cdot 4l = 4Pl = M_O(F_3) = M_{O3}(F_3) = 4Pl\bar{j}.$$

$\bar{F}_i, \bar{M}_i$	F_{ix}	F_{iy}	F_{iz}	M_x	M_y	M_z
F_1	0	-3Pl	4P	12Pl	0	0
F_2	2P	-3P	-4P	0	8Pl	-6Pl
F_3	P	0	0	0	4Pl	0
M	–	–	–	0	0	6Pl
Σ	3P	-6P	0	2Pl	12Pl	0





$$\overline{OA} \times \overline{R} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_A & y_A & z_A \\ R_x & R_y & R_z \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2l & 0 & 0 \\ 3P & -6P & 0 \end{vmatrix} = -12Pl\bar{k} \Rightarrow T = \bar{R} \cdot \bar{M}_A = -66P^2 l.$$

Cazul de reducere: torsor minimal.

$$\bar{M}_{min}=\frac{\bar{R}\cdot\bar{M}_O}{R^2}\bar{R}=\frac{-66P^2l}{45P^2}\left(3P\bar{i}-6P\bar{j}\right)=-\frac{22}{5}\left(Pl\bar{i}-2Pl\bar{j}\right)\Rightarrow\ T=\bar{R}\cdot\bar{M}_{min}=-66P^2l;$$

$$\Gamma_{min}=\begin{cases}\bar{R}=3P\bar{i}-6P\bar{j}\\ \bar{M}_{min}=-\dfrac{22}{5}\left(Pl\bar{i}-2Pl\bar{j}\right)\end{cases}$$

$$\frac{M_{Ox}-yR_z+zR_y}{R_x}=\frac{M_{Oy}-zR_x+xR_z}{R_y}=\frac{M_{Oz}-xR_y+yR_x}{R_z}\Rightarrow\ \frac{2l-6z}{3}=\frac{4l-z}{-2}=\frac{6x-3y}{0}\Rightarrow$$

$$\Rightarrow\begin{cases}2x+y=0\\ 12z-4l=12l-3z\end{cases}\Rightarrow\ z=\frac{16}{15}\cdot l;\begin{cases}x=1\Rightarrow\ y=-2\\ x=0\Rightarrow\ y=0\\ x=-1\Rightarrow\ y=2\end{cases}$$

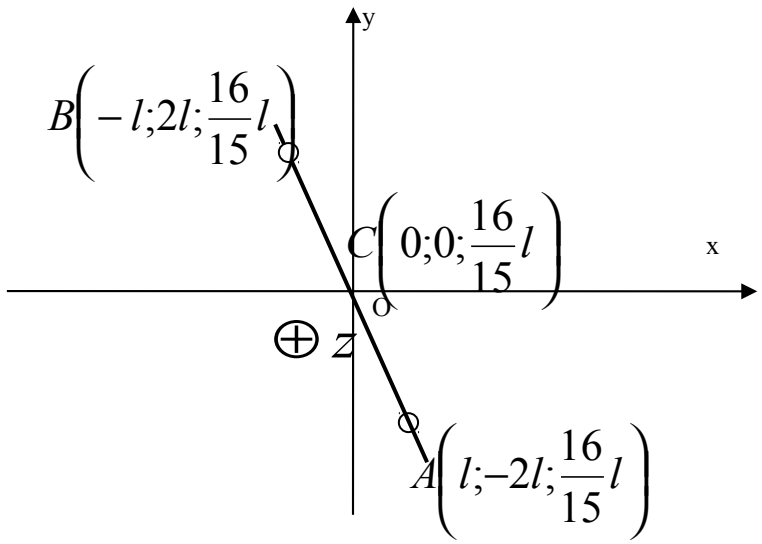


Fig. 3.3: Axa centrală

3.2.2. Probleme propuse

1)

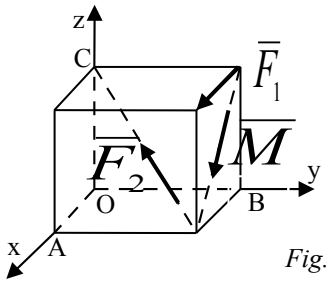
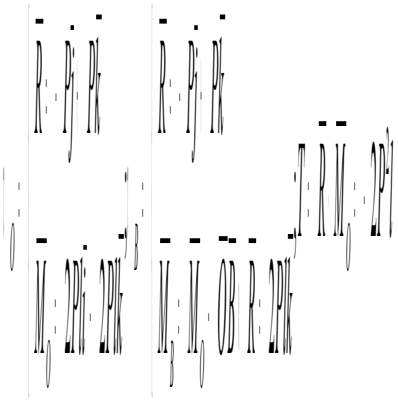


Fig. 3.4

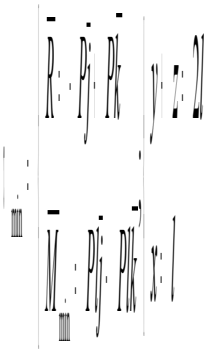
Date:

Se cer:
echivalența=?; axa centrală=?

Răspuns:



Cazul de reducere: torsor minimal.



2)

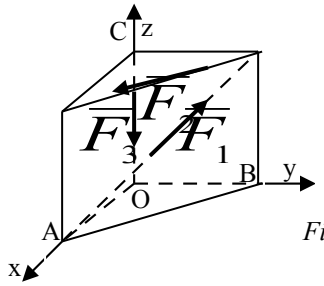
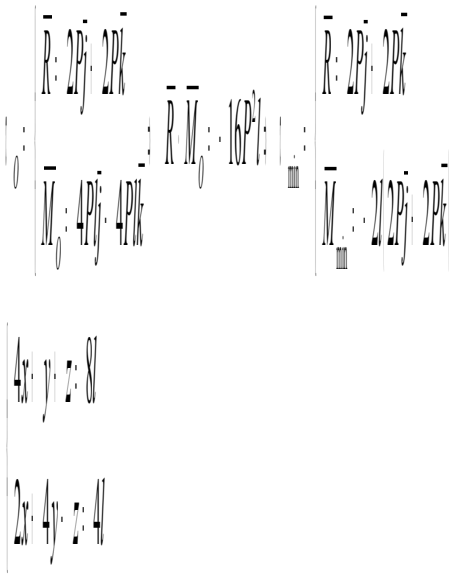


Fig. 3.5

Date:

Se cer:
echivalența=?; axa centrală=?

Răspuns:



3.3. Reducerea sistemelor de forțe coplanare

3.3.1. Probleme rezolvate

1)

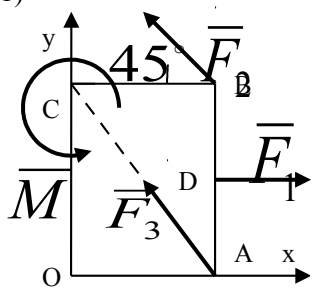


Fig. 3.6

Date:

Se cer:
echivalența=?; axa centrală=?

Rezolvare:

$$\overline{M}_O(\overline{F}_2) = \overline{OB} \times \overline{F}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_B & y_B & z_B \\ F_{2x} & F_{2y} & F_{2z} \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ l & 2l & 0 \\ -P & 2P & 0 \end{vmatrix} = 3Pl\bar{k};$$

$$\overline{F}_3 = F_3\bar{u}_3 = F_3\frac{\overline{AC}}{AC} = P\sqrt{5}\frac{(x_C - x_A)\cdot\bar{i} + (y_C - y_A)\cdot\bar{j}}{\sqrt{(x_C - x_A)^2 + (y_C - y_A)^2}} = P\bar{i} + 2P\bar{j};$$

$$\overline{M}_O(\overline{F}_3) = \overline{OA} \times \overline{F}_3 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_A & y_A & z_A \\ F_{3x} & F_{3y} & F_{3z} \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ l & 0 & 0 \\ -P & 2P & 0 \end{vmatrix} = 2Pl\bar{k}.$$

$\overline{F}_i, \overline{M}_i$	F_{ix}	F_{iy}	M_{iOz}
$\overline{F}_1$	2P	0	-2Pl

$\overline{F_2}$	-P	P	3Pl
$\overline{F_3}$	-P	2P	2Pl
$\overline{M}$	–	–	3Pl
Σ	0	3P	6Pl

$$\tau_O = \begin{cases} \overline{R} = 3P\bar{j} \\ \overline{M}_O = 6P\bar{k} \end{cases} \Rightarrow \overline{R} \cdot \overline{M}_O = 0;$$

$$\tau_A = \begin{cases} \overline{R} = 3P\bar{j} \\ \overline{M}_A = \overline{M}_O - \overline{OA} \times \overline{R} = 6P\bar{k} - \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_A & y_A & z_A \\ R_x & R_y & R_z \end{vmatrix} = 6P\bar{k} - \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ l & 0 & 0 \\ 0 & 3P & 0 \end{vmatrix} = 3P\bar{k} \end{cases} \Rightarrow \overline{R} \cdot \overline{M}_A = 0$$

Cazul de reducere: Rezultantă unică pe axa centrală

$$M_{Oz} - xR_y + yR_x = 0 \Rightarrow 6Pl - x \cdot 3P + y \cdot 0 = 0 \Rightarrow 6Pl - 3Px = 0 \Rightarrow 2l - x = 0 \Rightarrow x = 2l.$$

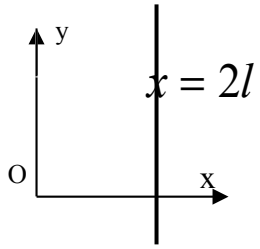


Fig.3.7: Axa centrală

3.3.2. Probleme propuse

1)

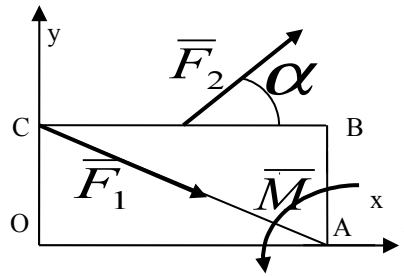


Fig. 3.8

Date:

Se cer:
echivalența=?; axa centrală=?

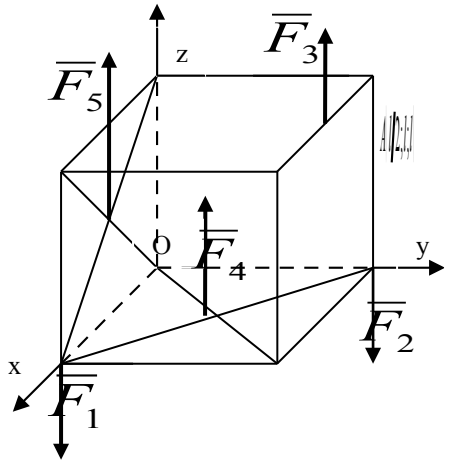
Răspuns:

$\overline{R} = 7P\overline{i}; \overline{M}_O = -7Pl\overline{k}; \overline{R} = 7P\overline{i}; \overline{M}_A = -7Pl\overline{k}; T = \overline{R} \cdot \overline{M}_O = 0; y = l;$

Cazul de reducere: Rezultantă unică pe axa centrală.

3.4. Reducerea sistemelor de forțe paralele

3.4.1. Probleme rezolvate



Date:
Cub rigid de muchie *l*;

Se cer:
echivalența=?; axa centrală=?;

Fig. 3.9

Rezolvare:

$\overline{F}_i$	F_{iz}	M_{ix}	M_{iy}
F_1	-P	0	Pl
F_2	-P	-Pl	0
F_3	2P	2Pl	-Pl
F_4	2P	Pl	-Pl
F_5	2P	0	-Pl
Σ	4P	2Pl	-2Pl

$$\begin{cases} \bar{R} = 4P\bar{k} \\ \bar{M}_O = 2Pl\bar{i} - 2Pl\bar{j} \end{cases} \Rightarrow \bar{R} \perp \bar{M}_O \Rightarrow 0.$$

Rezultantă unică pe axa centrală.

$$\frac{2Pl - 4Py}{0} = \frac{-2Pl + 4Px}{0} = \frac{0}{4P} \Rightarrow x = \frac{l}{2}; y = \frac{l}{2}.$$

$$x_C = \frac{\sum F_i x_i}{\sum F_i} = \frac{l}{2}; y_C = \frac{\sum F_i y_i}{\sum F_i} = \frac{l}{2}; z_C = \frac{\sum F_i z_i}{\sum F_i} = \frac{3l}{4}.$$

3.4.2. Probleme propuse

Date:

Se cer:

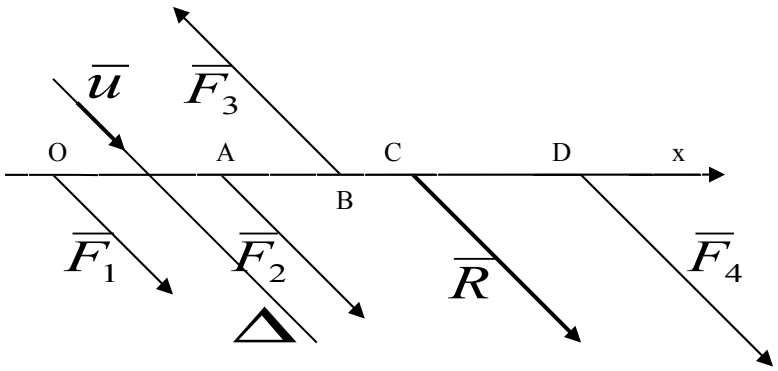


Fig. 3.10

Răspuns:

$\bar{R} = 10P\bar{u}; R = 10P; x_C = 5{,}9l.$

CAPITOLUL 4

CENTRE DE MASĂ (GREUTATE)

4.1. Generalități

4.1.1. Calculul coordonatelor. Centrul de masă pentru elemente geometrice cunoscute

Coordonatele centrului maselor se determină cu următoarele relații (în spațiu şi în plan):

a) pentru sisteme de puncte materiale:

$$x_C = \frac{\sum m_i x_i}{\sum m_i}; y_C = \frac{\sum m_i y_i}{\sum m_i}; z_C = \frac{\sum m_i z_i}{\sum m_i}$$

b) pentru linii materiale (bare) omogene:

$$x_C = \frac{\sum l_i x_i}{\sum l_i}; y_C = \frac{\sum l_i y_i}{\sum l_i}; z_C = \frac{\sum l_i z_i}{\sum l_i}$$

c) pentru suprafețe materiale (plăci) omogene:

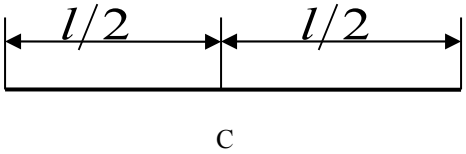
$$x_C = \frac{\sum A_i x_i}{\sum A_i}; y_C = \frac{\sum A_i y_i}{\sum A_i}; z_C = \frac{\sum A_i z_i}{\sum A_i}$$

d) pentru volume materiale (corpuri) omogene:

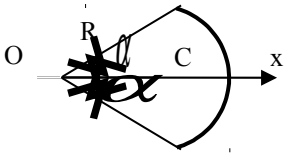
$$x_C = \frac{\sum V_i x_i}{\sum V_i}; y_C = \frac{\sum V_i y_i}{\sum V_i}; z_C = \frac{\sum V_i z_i}{\sum V_i}$$

Pentru diverse elemente geometrice cunoscute, centrul maselor se determină astfel:

a) Bară omogenă:

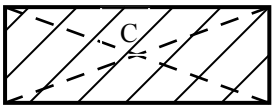


b) Bară în formă de sector circular omogen:

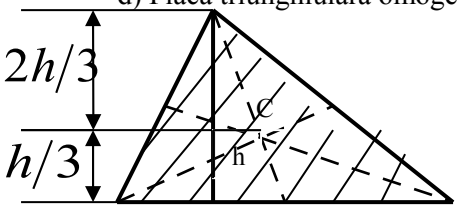


$$x_C = R\frac{\sin\alpha}{\alpha}; \alpha\left[radiani\right]$$

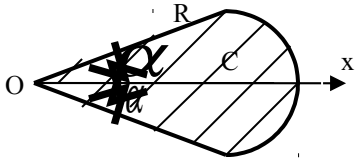
c) Placă dreptunghiulară omogenă:



d) Placă triunghiulară omogenă:



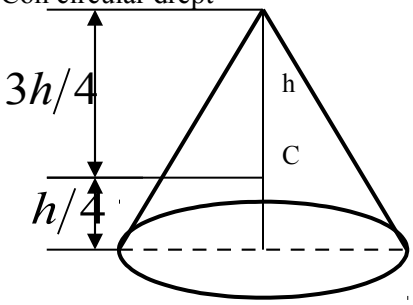
e) Placă în formă de sector circular omogen:



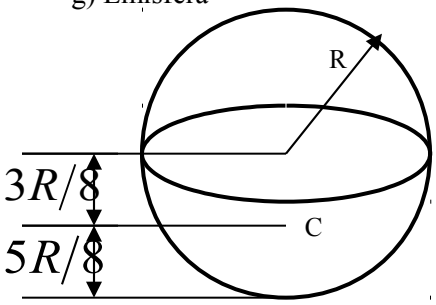
;

$$\alpha\left[radiani\right]$$

f) Con circular drept



g) Emisferă



4.1.2. Teoremele Pappus – Guldin

1) Aria suprafeței generată de un arc de curbă plană ce se roteşte în jurul unei axe din planul curbei

(arcul fiind situat integral de aceeaşi parte a axei), este egală cu lungimea arcului de curbă, multiplicată cu lungimea cercului descris de centrul de greutate al curbei date, presupuse omogenă.

$$A_{Ox} = 2\pi \cdot y_C \sum l_i ; A_{Oy} = 2\pi \cdot x_C \sum l_i .$$

2) Volumul corpului generat prin rotirea unei suprafeţe plane închise, în jurul unei axe din planul ei (suprafaţa fiind integral situată de aceeaşi parte a axei), este egal cu produsul dintre aria acestei suprafeţe şi lungimea cercului descris de centrul ei de greutate(arie considerată omogenă).

$$V_{Ox} = 2\pi \cdot y_C \sum A_i ; V_{Oy} = 2\pi \cdot x_C \sum A_i .$$

4.2. Centrul maselor pentru un sistem de puncte materiale

4.2.1. Probleme rezolvate

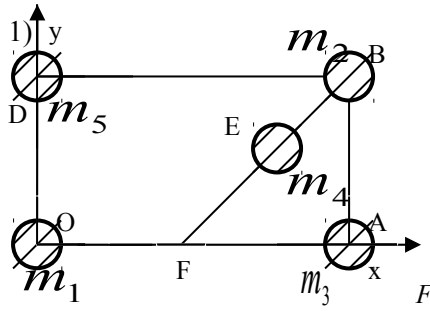


Fig. 4.1.

Date:

Se cer:

Rezolvare:

Punctul i	m_i	x_i	y_i	$m_i x_i$	$m_i y_i$
1	*m*	*0*	*0*	*0*	*0*
2	*2m*	*2l*	*l*	*4ml*	*2ml*
3	*3m*	*2l*	*0*	*6ml*	*0*
4	*4m*	*1,5l*	*0,5l*	*6ml*	*2ml*
5	*5m*	*0*	*l*	*0*	*5ml*
∑	*15m*	-	-	*16ml*	*9ml*

$$x_C = \frac{\sum m_i x_i}{\sum m_i} = \frac{16ml}{15m} = 1{,}07l ; y_C = \frac{\sum m_i y_i}{\sum m_i} = \frac{9ml}{15m} = 0{,}6l.$$

4.2.2. Probleme propuse

1)

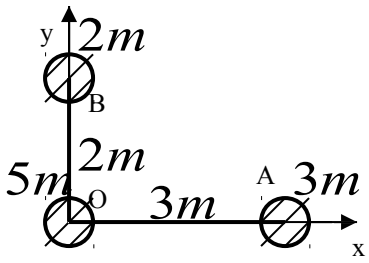


Fig. 4.2

Date:

Răspuns:

Se cer:

2)

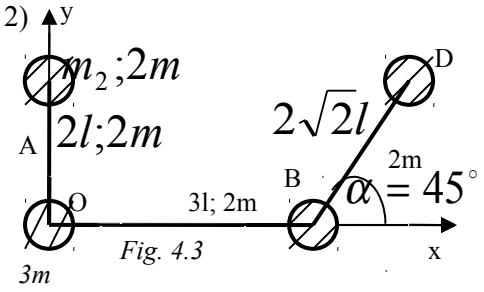


Fig. 4.3

Date:

;

A(3m); O(5m); B(3m); D(3m).

BD(); $\alpha = 45^0$

Se cer:

Răspuns:

$$x_C = \frac{35}{20}l = 1{,}75l; y_C = \frac{4}{5}l = 0{,}8l.$$

4.3. Centrul maselor pentru linii materiale

4.3.1 Probleme rezolvate

1)

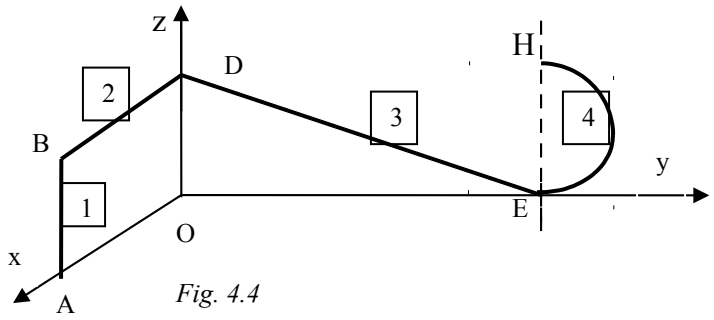


Fig. 4.4

Date:

Se cer:

Rezolvare:

Bara	l_i	x_i	y_i	z_i	$l_i x_i$	$l_i y_i$	$l_i z_i$
1	l	l	0	$l/2$	$l/2$	0	$l^2/2$
2	l	$l/2$	0	l	l	0	l^2

3	$l\sqrt{5}$	0	l	$l/2$	0	$l^2\sqrt{5}$	$\frac{l^2}{2}\sqrt{5}$
4	$l/2$	0	$\frac{l}{\pi}\left(2\pi+\sqrt{2}\right)$	$l/2$	0	$\frac{l^2}{2}\left(2\pi+\sqrt{2}\right)$	$\frac{\pi\cdot l^2}{4}$
Σ	$\frac{l}{2}\left(4+\sqrt{5}+\pi\right)$	–	–	–	$\frac{3l^2}{2}$	$\frac{l^2}{2}\left(2\pi+\sqrt{2}+2\sqrt{5}\right)$	$\frac{l^2}{4}\left(6+2\sqrt{5}+\pi\right)$

$$x_C=\frac{\sum l_i x_i}{\sum l_i}=\frac{3l}{6+\sqrt{5}+\pi};y_C=\frac{2\pi+2\sqrt{5}+\sqrt{2}}{4+\pi+\sqrt{5}}l;z_C=\frac{l}{2}\cdot\frac{6+2\sqrt{5}+\pi}{4+\sqrt{5}+\pi}.$$

2)

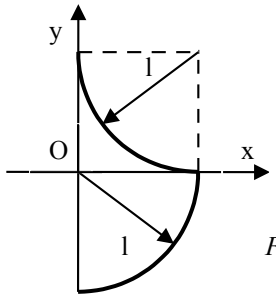
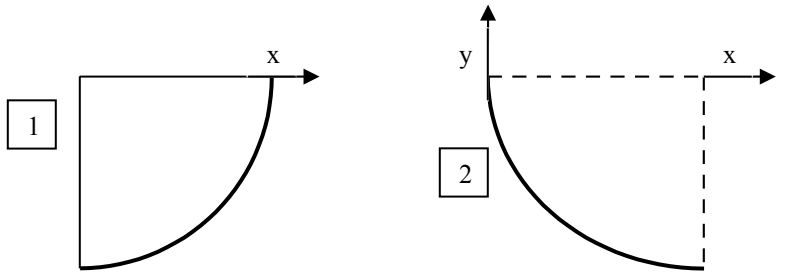


Fig. 4.5

Date:
Bară plană omogenă; l.

Se cer:

Rezolvare:



Bara	l_i	x_i	y_i	$l_i x_i$	$l_i y_i$
1	$\frac{\pi\cdot l}{2}$	$\frac{2\cdot l}{\pi}$	$-\frac{2\cdot l}{\pi}$	l^2	$-l^2$
2	$\frac{\pi\cdot l}{2}$	$\frac{l}{\pi}(\pi-2)$	$\frac{l}{\pi}(\pi-2)$	$\frac{l^2}{2}(\pi-2)$	$\frac{l^2}{2}(\pi-2)$

Σ	$\pi \cdot l$	–	–	$\frac{l^2}{2} \cdot \pi$	$\frac{l^2}{2}(\pi - 4)$

$$x_C = \frac{\sum l_i x_i}{\sum l_i} = \frac{l}{2}; y_C = \frac{\sum l_i y_i}{\sum l_i} = \frac{\pi - 4}{2 \cdot \pi} l; A_{Oy} = 2 \cdot \pi \cdot x_C \sum l_i = \pi^2 l^2.$$

Determinarea ariei A_{Ox} nu are sens deoarece bara nu se găseşte integral de aceeaşi parte a axei Ox.

4.3.2. Probleme propuse

1)

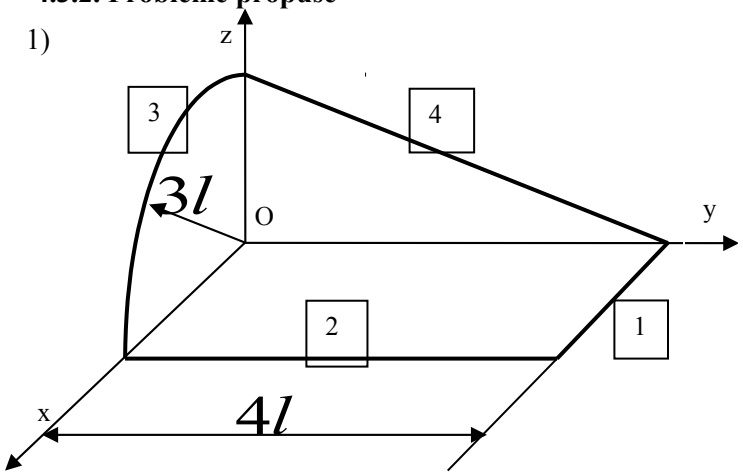


Fig. 4.6:

Date:
Bară omogenă; l.

Se cer:

Răspuns:

2)

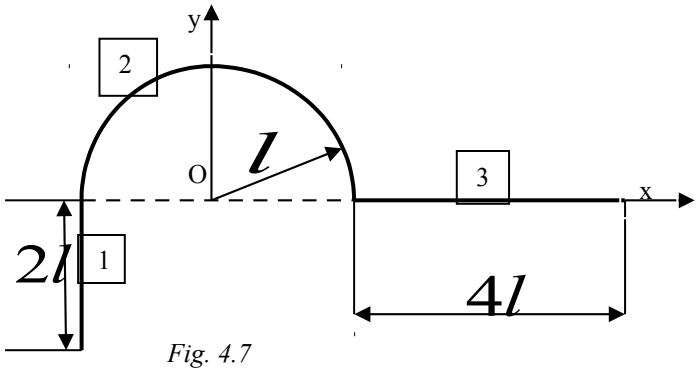


Fig. 4.7

Date:
Bară plană omogenă; l.

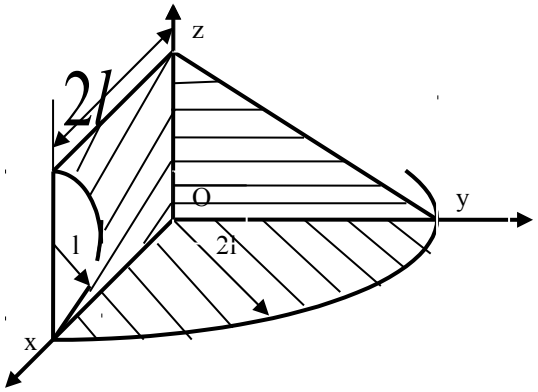
Se cer:

Răspuns:

4.4. Centrul maselor pentru suprafețe materiale

4.4.1 Probleme rezolvate

1)

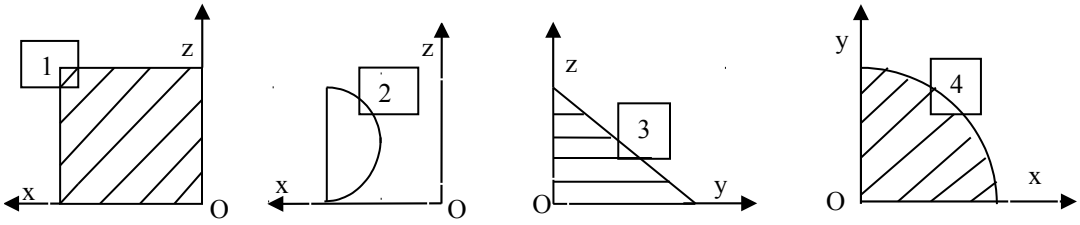


Date:
Placă omogenă; l.

Se cer?

Fig. 4.8

Rezolvare:



Placa	A_i	x_i	y_i	z_i	$A_i x_i$	$A_i y_i$	$A_i z_i$
1	$4l^2$	l	0	l	$4l^3$	0	$4l^3$
2	$-\frac{\pi \cdot l^2}{2}$	$\frac{2l}{3\pi}(3\pi-2)$	0	l	$\frac{3\pi-2}{3}l^3$	0	$-\frac{\pi \cdot l^3}{2}$
3	$2l^2$	0	$\frac{2l}{3}$	$\frac{2l}{3}$	0	$\frac{4l^3}{3}$	$\frac{4l^3}{3}$
4	$\pi \cdot l^2$	$\frac{8l}{3\pi}$	$\frac{8l}{3\pi}$	0	$\frac{8l^3}{3}$	$\frac{8l^3}{3}$	0
Σ	$\frac{12+\pi}{2} \cdot l^2$	–	–	–	$(6+\pi) \cdot l^3$	$4l^3$	$\frac{32-\pi}{6}l^3$

$$x_C = \frac{\sum A_i x_i}{\sum A_i} = \frac{12+2\pi}{12+\pi} \cdot l; y_C = \frac{\sum A_i y_i}{\sum A_i} = \frac{8l}{12+\pi}; z_C = \frac{\sum A_i z_i}{\sum A_i} = \frac{32-3\pi}{12+\pi} \cdot l.$$

2)

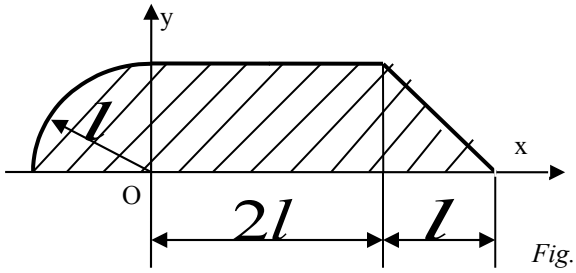
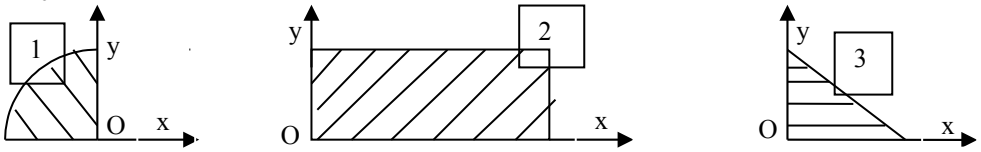


Fig. 4.9

Date:
Placă plană omogenă; l.

Se cer:

Rezolvare:



Placa	A_i	x_i	y_i	$A_i x_i$	$A_i y_i$
1	$\frac{\pi}{4}\cdot l^2$	$-\frac{4l}{3\pi}$	$\frac{4l}{3\pi}$	$-\frac{l^3}{3}$	$\frac{l^3}{3}$
2	$2l^2$	l	$\frac{l}{2}$	$2l^3$	l^3
3	$\frac{l^2}{2}$	$\frac{7l}{3}$	$\frac{l}{3}$	$\frac{7l^3}{6}$	$\frac{l^3}{6}$
Σ	$\frac{10+\pi}{4}l^2$	–	–	$\frac{17l^3}{6}$	$\frac{3l^3}{2}$

$$x_C=\frac{\sum A_i x_i}{\sum A_i}=\frac{34}{3}\cdot\frac{l}{10+\pi};y_C=\frac{\sum A_i y_i}{\sum A_i}=\frac{6l}{10+\pi};V_{Ox}=2\pi\cdot y_C\sum A_i=3\pi\cdot l^3.$$

Determinarea volumului V_{Oy} nu are sens deoarece placa nu se găseşte integral de aceeaşi parte a axei *Oy*.

4.4.2. Probleme propuse

1)

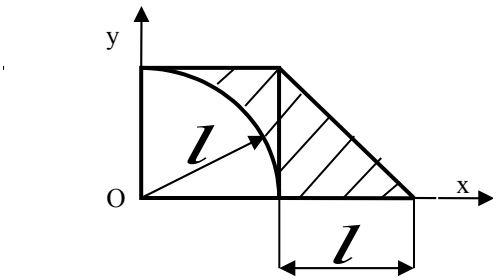


Fig. 4.10

Date:
Placă plană omogenă; l.

Se cer:

Răspuns:

$$x_C = \frac{10 \cdot l}{3 \cdot (6-\pi)}; y_C = \frac{4 \cdot l}{6-\pi}; V_{Ox} = 2\pi \cdot l^3; V_{Oy} = \frac{5}{3}\pi \cdot l^3.$$

2)

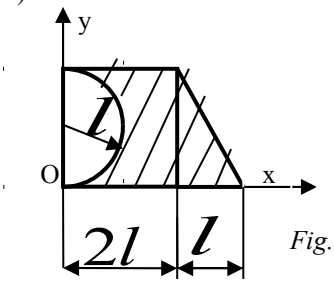


Fig. 4.11

Date:
Placă plană omogenă; l.

Se cer:

Răspuns:

4.5. Centrul maselor pentru volume materiale

4.5.1 Probleme rezolvate

1)

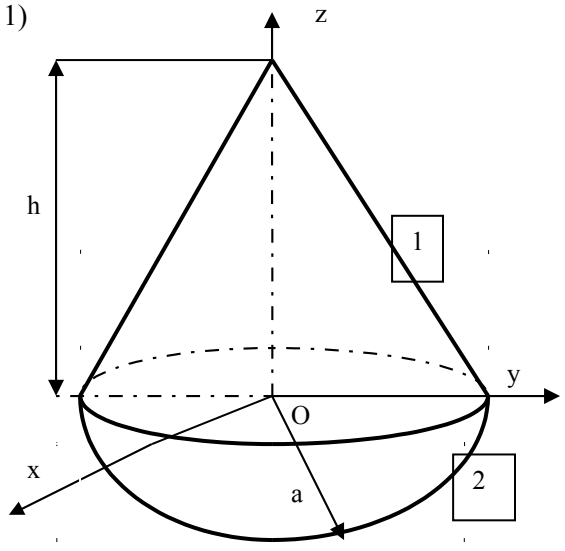


Fig. 4.12

Date:
Corp omogen; a

Se cer:
astfel încât

Observație:

Rezolvare:

Corp	V_i	z_i	$V_i z_i$
1	$\frac{\pi \cdot a^2 h}{3}$	$\frac{h}{4}$	$\frac{\pi \cdot a^2 h^2}{12}$
2	$\frac{2\pi \cdot a^3}{3}$	$-\frac{3}{8}a$	$-\frac{\pi \cdot a^4}{4}$
Σ	$\frac{\pi \cdot a^2}{3}(h+2a)$	–	$\frac{\pi \cdot a^2}{12}(h^2-3a^2)$

$$z_C = \frac{\sum V_i z_i}{\sum V_i} = \frac{h^2 - 3a^2}{4(h+2a)};$$

$$C \equiv O \Rightarrow z_C = 0 \Rightarrow \frac{h^2 - 3a^2}{4(h+2a)^2} = 0 \Rightarrow h^2 - 3a^2 = 0 \Rightarrow h^2 = 3a^2 \Rightarrow h = \pm a\sqrt{3} \Rightarrow h = a\sqrt{3}.$$

4.5.2. Probleme propuse

2)

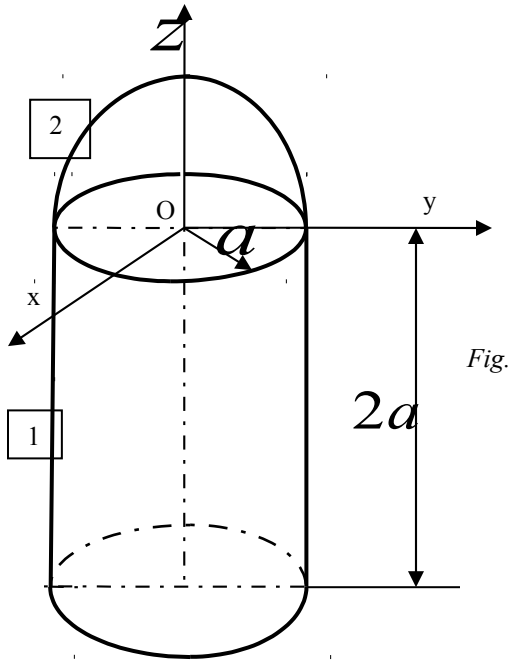


Fig. 4.13

Date:
Corp omogen; a

Se cere:

Răspuns:

CAPITOLUL 5

STATICA RIGIDULUI

5.1. Generalități

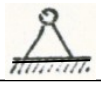
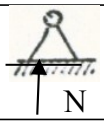
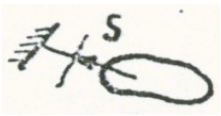
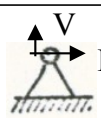
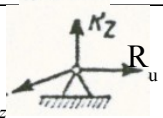
Condiția necesară şi suficientă ca rigidul liber să fie în echilibru:

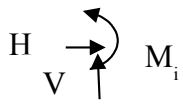
$$\begin{cases} \overline{R} = \sum_{i=1}^{n} \overline{F}_i = 0 \\ \overline{M}_0 = \sum_{i=1}^{n} \overline{M}_{0i} = \sum_{i=1}^{n} \overline{OA}_i \times \overline{F}_i = 0 \end{cases}$$

În raport cu un sistem de referință considerat se scriu şase ecuații scalare de proiecții:

$$\begin{aligned} R_x &= \sum F_{ix} = 0 \\ R_y &= \sum F_{iy} = 0 \\ R_z &= \sum F_{iz} = 0 \\ M_{Ox} &= \sum M_{Oxi} = 0 \\ M_{Oy} &= \sum M_{Oyi} = 0 \\ M_{Oz} &= \sum M_{Ozi} = 0 \end{aligned}$$

Principalele legături ale solidului rigid sunt: reazemul simplu, articulația, încastrarea şi prinderea cu fire în care se introduc următoarele reacțiuni conform tabelului de mai jos:

Legatura		Simbol	Cu ce se inlocuieste	Nr necunoscuteleor pe care le introduce
Reazemul simplu				1
Prinderea cu fire		———		1
Articulatia	Plana			2
	Sferica			3



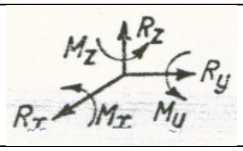
Incastrarea	Plana			3
	Spatiala			6

Fig. 5.1

Frecările care apar în legăturile solidului rigid sunt de mai multe feluri şi anume:

a) frecare de alunecare caracterizată de relaţia:

$$F_f \leq \mu \cdot N$$

b) frecare de rostogolire caracterizată de relaţia:

$$M_r \leq s \cdot N$$

c) frecarea de pivotare caracterizată de relaţia:

$$M_p \leq \frac{2}{3}\mu \cdot \frac{R^3 - r^3}{R^2 - r^2} \cdot N$$

d) frecarea în articulaţii şi lagăre caracterizată de relaţia:

$$M_f \leq \mu_0 \cdot r_0 \sqrt{H^2 + V^2}$$

e) frecarea firelor caracterizată de formulele lui Euler reunite:

$$F_r e^{-\mu \cdot \theta} \leq F_m \leq F_r e^{\mu \cdot \theta}$$

5.2. Echilibrul rigidului cu legături fără frecare

5.2.1. Probleme rezolvate

1)

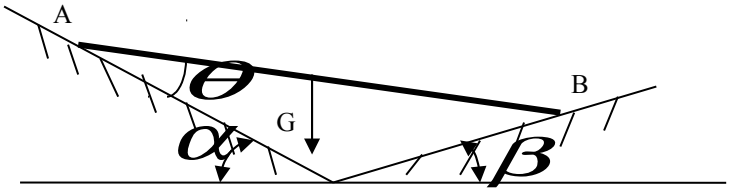
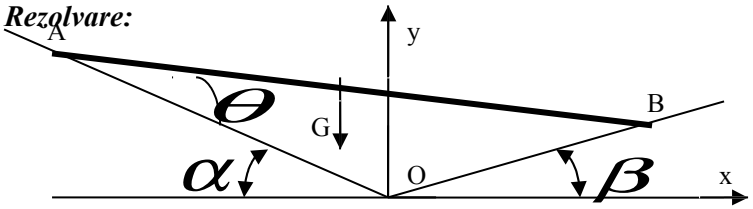


Fig. 5.2

Date:

Se cer:
pentru echilibru;

Rezolvare:



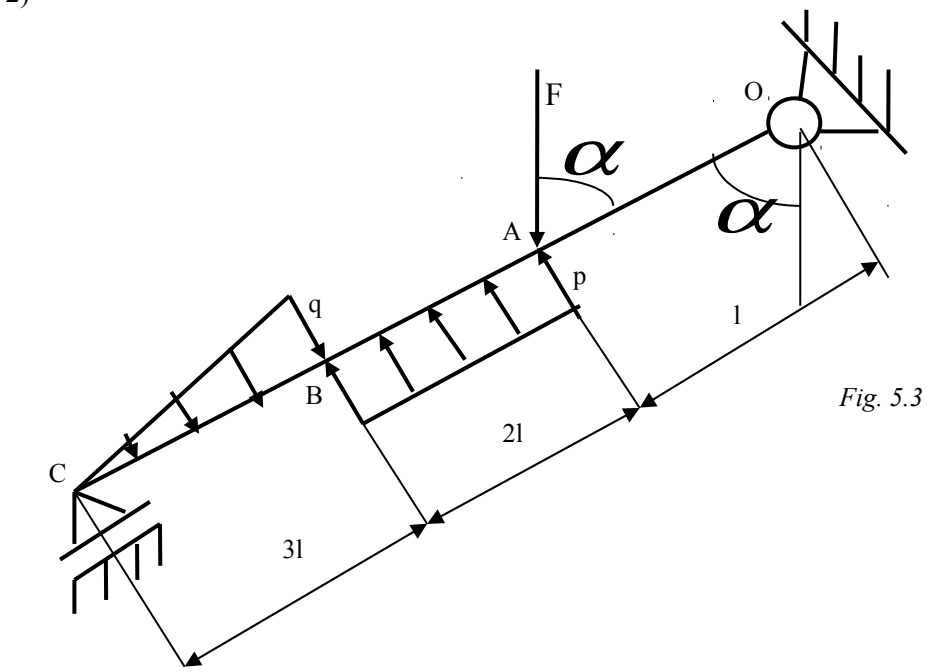
$$\sum F_{ix} = 0 : N_A \sin\alpha - N_B \sin\beta = 0$$
$$\sum F_{iy} = 0 : N_A \cos\alpha + N_B \cos\beta - G = 0$$
$$\sum M_{iA} = 0 : -Gl\cos(\alpha-\theta) + N_B \cdot 2l \cdot \cos(\alpha+\beta-\theta) = 0$$

$$N_A = \frac{G\sin\beta}{\sin(\alpha+\beta)}; N_B = \frac{G\sin\alpha}{\sin(\alpha+\beta)}; tg\theta = ctg\alpha - 2ctg(\alpha+\beta).$$

Observație:

$\alpha=\beta=\theta$, rezultă că bara este orizontală

2)

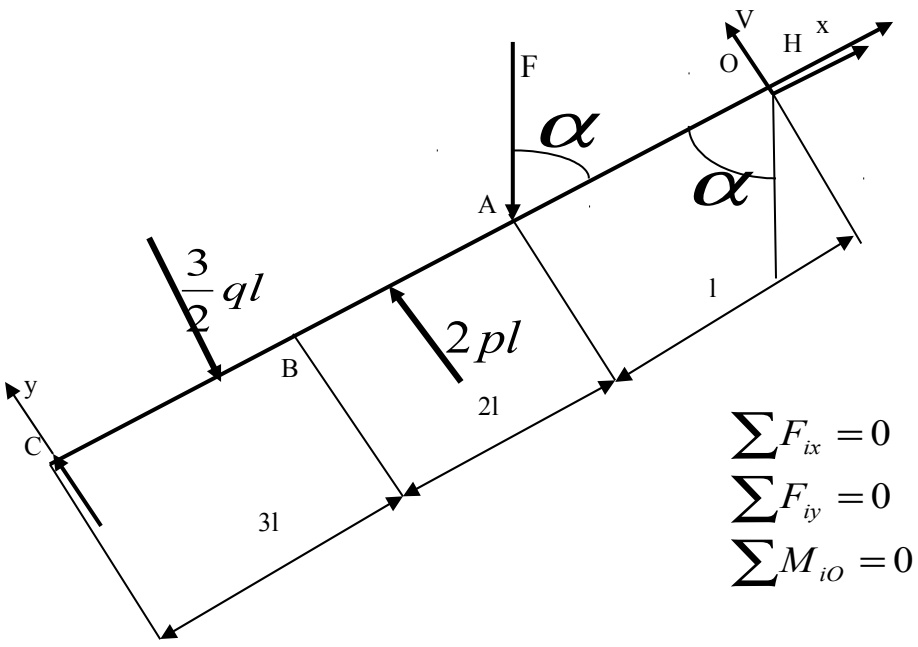


Date:

Se cer:
Reacțiuni în O şi C.

Fig. 5.3

Rezolvare:



$$\sum F_{ix} = 0$$
$$\sum F_{iy} = 0$$
$$\sum M_{iO} = 0$$

$$H - F\cos\alpha = 0$$

$$2pl + V - F\sin\alpha - \frac{3}{2}ql + N = 0$$

$$Fl\sin\alpha - 2pl \cdot 2l + \frac{3}{2}ql \cdot 4l - N \cdot 6l = 0$$

$$H = F\cos\alpha$$

$$N = \frac{F\sin\alpha}{6} - \frac{2}{3}pl + ql$$

$$V = F\sin\alpha - 2pl + \frac{3}{2}ql - \frac{F\sin\alpha}{6} + \frac{2}{3}ql - ql = \frac{5}{6}F\sin\alpha - 2pl + \frac{13}{6}ql$$

3)

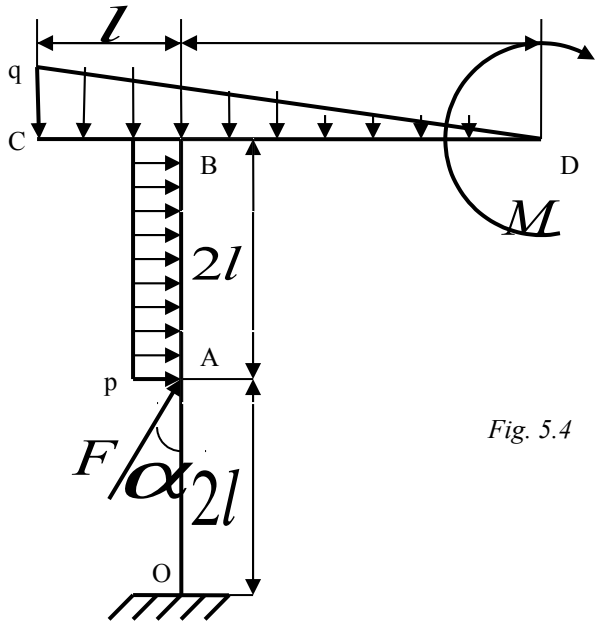
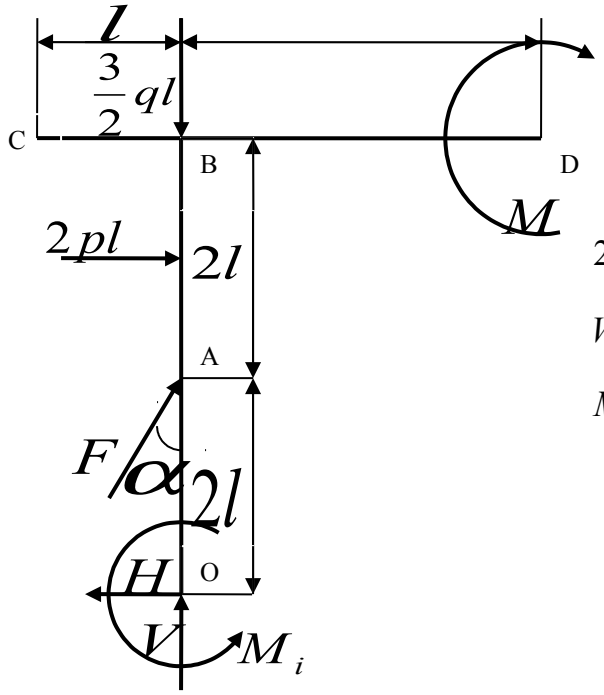


Fig. 5.4

Date:

Se cer:
Reacţiuni în O

Rezolvare:



$$\sum F_{ix} = 0:$$
$$\sum F_{iy} = 0:$$
$$\sum M_{iO} = 0$$

$$2pl + F\sin\alpha - H = 0$$

$$V + F\cos\alpha - \frac{3}{2}ql = 0$$

$$M_i - F\sin\alpha \cdot 2l - 2pl \cdot 3l - M = 0$$

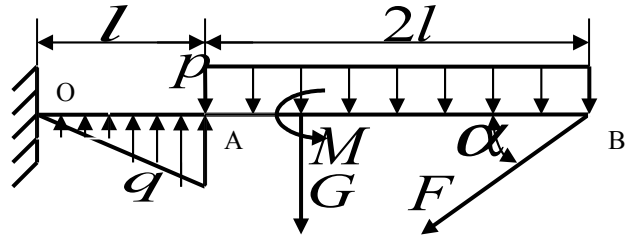
$$H = 2pl + F\sin\alpha$$

$$V = \frac{3}{2}ql - F\cos\alpha$$

$$M_i = 2Fl\sin\alpha + 6pl^2 + M$$

5.2.2. Probleme propuse

1)



Date:

Se cer:
Reacțiuni în O.

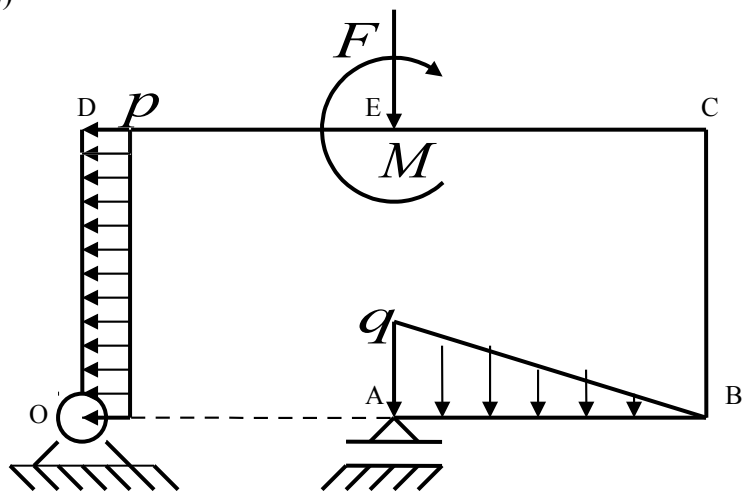
Fig. 5.5

Răspuns:

$$H_O = F\cos\alpha; V_O = G + F\sin\alpha + 2pl - \frac{ql}{2};$$

$$M_O = \frac{3}{2}Gl - \frac{ql}{2}\cdot\frac{2}{3}l - M + 2pl\cdot 2l + F\sin\alpha\cdot 3l = \frac{3}{2}Gl - \frac{ql^2}{3} - M + 4pl^2 + 3Fl\sin\alpha.$$

2)



Date:

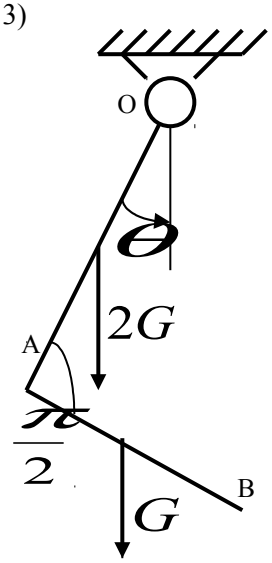
Se cer:
Reacțiunile din O şi A.

Fig. 5.6

Răspuns:

$$H_O = pl, V_O = \frac{pl}{2} - \frac{M}{l} - \frac{ql}{4}, N_A = F + \frac{M}{l} + \frac{3}{4}ql - \frac{pl}{2}.$$

3)



Date:

Se cer:
Reacțiuni în O.

Răspuns:

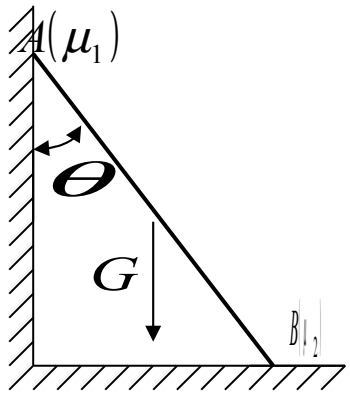
Fig. 5.7

5.3. Statica rigidului cu legături cu frecare

5.3.1. Frecarea de alunecare

5.3.1.1. Probleme rezolvate

1)

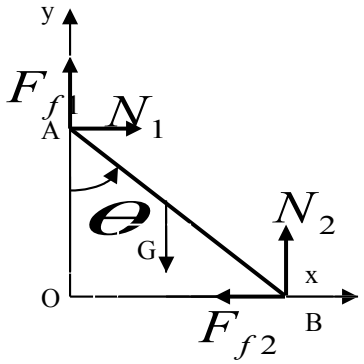


Date:

Se cere:
Poziția de echilibru a barei:

Fig. 5.8

Rezolvare:



$$\sum F_{ix} = 0$$
$$\sum F_{iy} = 0$$
$$\sum M_A = 0$$

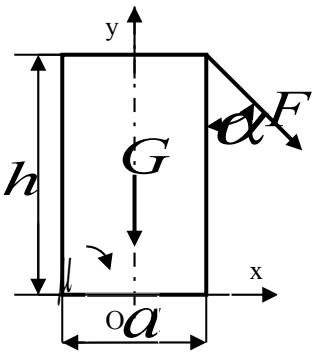
Se alege N_2 ca necunoscută intermediară şi se obține:

$$F_{f1} = G - N_2, N_1 = F_{f2} = N_2 tg\theta - \frac{G}{2} tg\theta$$

Înlocuind în cele două inecuații rezultă:

$$N_2 \geq \frac{G + \frac{1}{2}\mu_1 G \cdot tg\theta}{1 + \mu_1 tg\theta}; N_2 \leq \frac{\frac{G}{2} tg\theta}{tg\theta - \mu_2} \Rightarrow \frac{2 + \mu_1 tg\theta}{1 + \mu_1 tg\theta} \leq \frac{tg\theta}{tg\theta - \mu_2} \Rightarrow tg\theta \leq \frac{2\mu_2}{1 - \mu_1\mu_2}.$$

2)



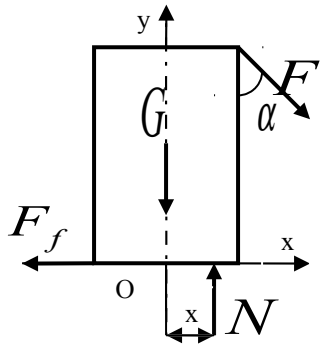
Date:

Se cer:
pentru echilibru

Fig. 5.9

Rezolvare:

Stricarea echilibrului apare din cauza tendinţei de alunecare şi din cauza tendinţei de răsturnare.



$$\sum F_{ix} = 0:$$
$$\sum F_{iy} = 0:$$
$$\sum M_{iO} = 0:$$

$$F \le \frac{\mu \cdot G}{\sin\alpha - \mu \cdot \cos\alpha}; a \ge \frac{2F\sin\alpha}{G} h.$$

5.3.1.2. Probleme propuse

1)

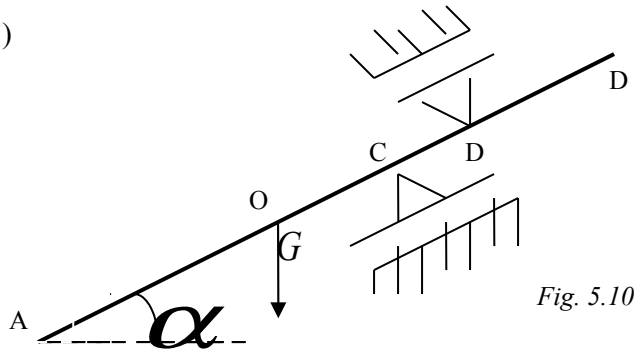


Fig. 5.10

Date:

În A bara nu se sprijină.

Se cere:
Pentru echilibrul barei.

Răspuns:

$$\frac{a(\sin\alpha - \mu \cdot \cos\alpha)}{2\mu \cdot \cos\alpha} \le x < l - a.$$

2)

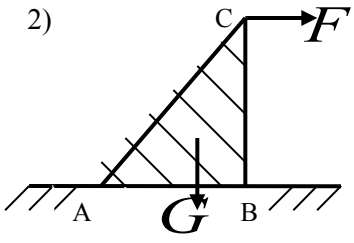


Fig. 5.11

Date:
A, B – legături bilatere; placă plană omogenă ABC; G; AB = a; BC = h;

Se cere:
pentru echilibru. Discuţie.

Răspuns:

$$a \ge 3\mu \cdot h \Rightarrow F \le \mu \cdot G;$$

$$2\mu \cdot h < a < 3\mu \cdot h \Rightarrow F \le \frac{\mu \cdot a \cdot G}{3(a - 2\mu \cdot h)}$$

5.3.2. Frecarea de rostogolire

5.3.2.1. Probleme rezolvate

1)

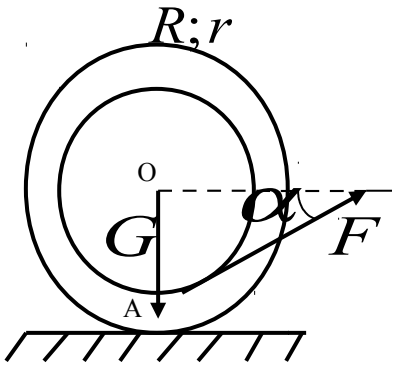
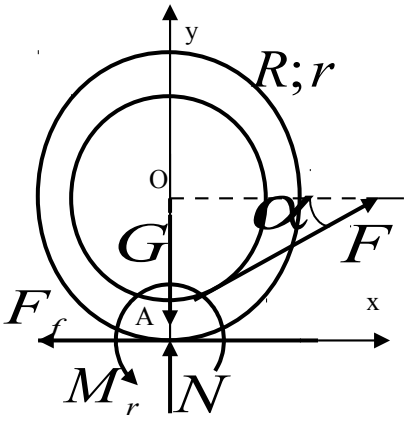


Fig. 5.12

Date:

Se cer:
pentru echilibru.

Rezolvare:



$$\sum F_{ix} = 0 \qquad F\cos\alpha - F_f = 0$$
$$\sum F_{iy} = 0 \qquad N - G + F\sin\alpha = 0$$
$$\sum M_{iO} = 0 \qquad M_r + Fr - F_f R = 0$$
$$alunecare: F_f \le \mu \cdot N$$
$$rostogolire: M_r \le s \cdot N$$

Rezultă:

$$F \le \frac{\mu \cdot G}{\cos\alpha + \mu \cdot \sin\alpha} = F_{alunecare}; F \le \frac{s \cdot G}{R\cos\alpha - r + s \cdot \sin\alpha} = F_{rostogolire} \Rightarrow$$
$$\Rightarrow F \le \min\left(F_{alunecare}; F_{rostogolire}\right)$$

2)

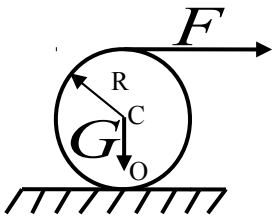


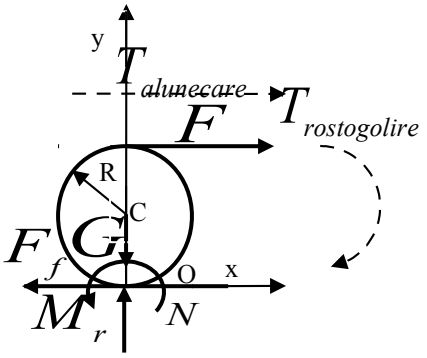
Fig. 5.13

Date:

Se cer:
pentru echilibru.

Rezolvare:

Se scriu ecuațiile de proiecții şi de momente.



$$\sum F_{ix} = 0 \quad F - F_f = 0$$
$$\sum F_{iy} = 0 \quad N - G = 0$$
$$\sum M_{iO} = 0 \quad M_r - 2FR = 0$$
$$alunecare: F_f \le \mu \cdot N$$
$$rostogolire: M_r \le s \cdot N$$

Rezultă: $F \le \mu \cdot G,$ *să nu alunece*

$F \le \frac{s}{2R} \cdot G,$ *să nu se rostogolească*

$$F \le \min\left(\mu \cdot G; \frac{s}{2R} \cdot G\right)$$

5.3.2.2. Probleme propuse

1)

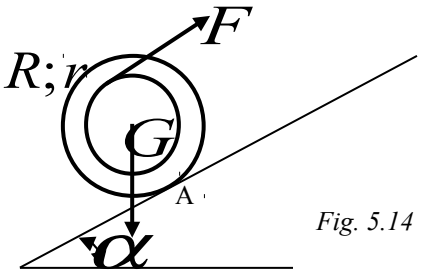


Fig. 5.14

Date:

Se cer:
pentru echilibru.

Răspuns:

Să nu alunece:

Să nu se rostogolească: $\frac{G(R\sin\alpha - s\cdot\cos\alpha)}{R+r} \le F \le \frac{G(R\cos\alpha + s\cdot\sin\alpha)}{R+r}$

2)

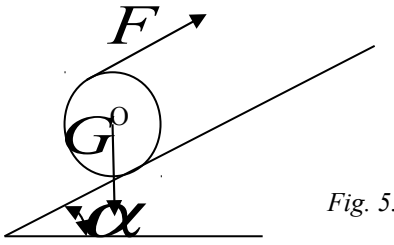


Fig. 5.15

Date:

Se cer:
pentru echilibru.

Răspuns:

Să nu alunece: $G(\sin\alpha - \mu\cdot\cos\alpha) \le F \le G(\sin\alpha + \mu\cdot\cos\alpha)$

Să nu se rostogolească: $\frac{G}{2}\left(\sin\alpha - \frac{s}{R}\cos\alpha\right) \le F \le \frac{G}{2}\left(\sin\alpha + \frac{s}{R}\cos\alpha\right)$

5.3.3. Frecarea în articulaţii şi lagăre

5.3.3.1. Probleme rezolvate

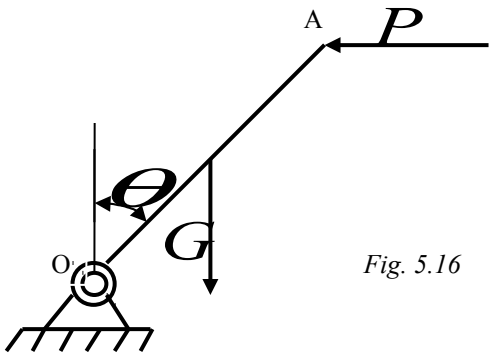
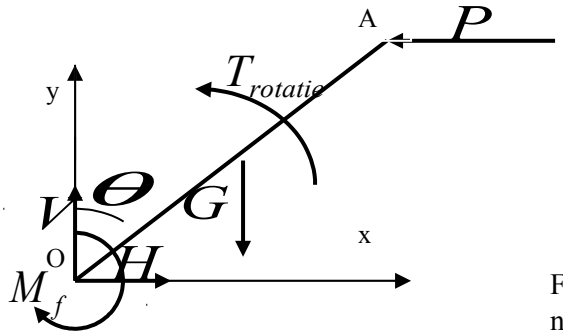


Fig. 5.16

Date:

Se cere:
(poziţia de echilibru).

Rezolvare:



Fie poziţia de echilibru a barei în care se neglijează frecarea în articulaţie adică .

$$\sum M_{iO} = 0 : Pl\cos\theta_0 - G\frac{l}{2}\sin\theta_0 = 0 \Rightarrow tg\theta_0 = \frac{2P}{G}$$

Se amplifică cu şi rezultă:

$$Pl\cos\theta - G\frac{l}{2}\sin\theta \le \mu_0 r_0\sqrt{P^2+G^2}$$

Dar

$$\frac{2P}{G}\cos\theta - \sin\theta \le \frac{2\mu_0 r_0}{Gl}\sqrt{P^2+G^2}$$

$$\cos\theta\cdot tg\theta_0 - \sin\theta \le \frac{2\mu_0 r_0}{Gl}\sqrt{P^2+G^2} \text{ sau } \sin(\theta_0-\theta) \le \frac{2\mu_0 r_0}{l}\cos\theta_0\sqrt{1+\frac{tg^2\theta_0}{4}}$$

Dacă membrul drept este subunitar atunci se notează:

În caz contrar inegalitatea anterioară este satisfăcută pentru orice unghi θ. Rezultă:

$$\sin(\theta_0-\theta) \le \sin\varphi \Rightarrow \theta_0-\theta \le \varphi \Leftrightarrow \theta_0-\varphi \le \theta.$$

Dacă tendinţa de rotaţie este în sens orar atunci se obţine soluţia completă şi anume:

$$\theta_0-\varphi \le \theta \le \theta_0+\varphi$$

unde:

$$\varphi = \arcsin\left(\frac{2\mu_0 r_0}{l} \cos\theta_0 \sqrt{1 + \frac{tg^2\theta_0}{4}} \right)$$

5.3.3.2. Probleme propuse

1)

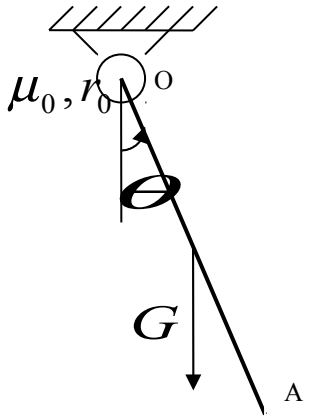


Fig. 5.17

Date:

Se cere:
la echilibru.

Răspuns:

2)

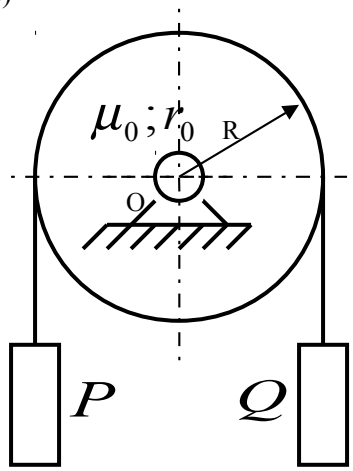


Fig. 5.18

Date:

Se cere:
la echilibru.

Răspuns:

5.3.4. Frecarea firelor

5.3.4.1. Probleme rezolvate

1)

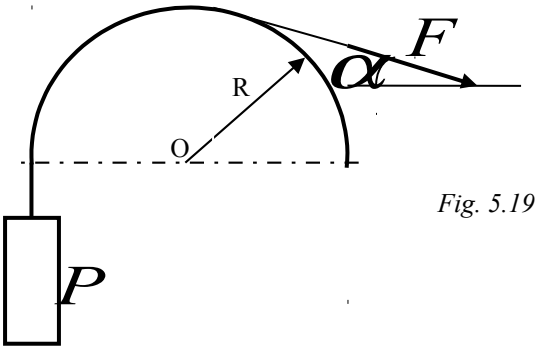
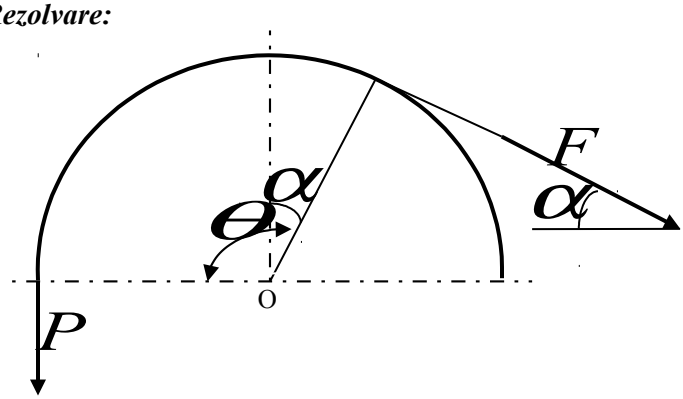


Fig. 5.19

Date:

Se cere:
pentru echilibru.

Rezolvare:



$$F_m \le F_r e^{\mu\cdot\theta};\theta=\frac{\pi}{2}+\alpha.$$

$$Pe^{-\mu\cdot\theta} \le F \le Pe^{\mu\cdot\theta} \Leftrightarrow$$

$$\Leftrightarrow Pe^{-\mu\cdot\left(\frac{\pi}{2}+\alpha\right)} \le F \le Pe^{\mu\cdot\left(\frac{\pi}{2}+\alpha\right)}.$$

5.3.4.2. Probleme propuse

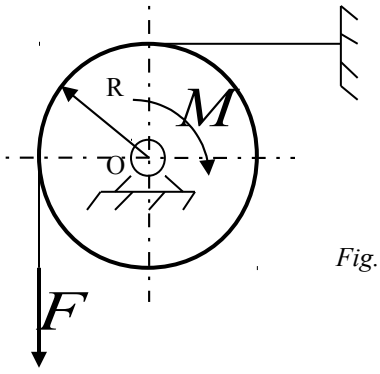


Fig. 5.20

Date:

Se cer:
pentru echilibru;
pentru echilibru
când îşi schimbă sensul.

Răspuns:

$$F \geq \frac{M}{R} \cdot \frac{e^{\mu \cdot \frac{\pi}{2}}}{e^{\mu \cdot \frac{\pi}{2}} - 1}; F^* \geq \frac{M}{R} \cdot \left(e^{\mu \cdot \frac{\pi}{2}} - 1 \right).$$

CAPITOLUL 6

STATICA SISTEMELOR

6.1. Generalități

6.1.1. Metoda izolării corpurilor

$$\begin{aligned} &\overline{F}_1 + \overline{F}_{12} + \ldots + \overline{F}_{1n} = 0 \\ &\overline{F}_2 + \overline{F}_{21} + \ldots + \overline{F}_{2n} = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\overline{F}_n + \overline{F}_{n1} + \ldots + \overline{F}_{nn} = 0 \end{aligned} \qquad \begin{aligned} &\overline{r}_1 \times \overline{F}_1 + \overline{r}_1 \times \overline{F}_{12} + \ldots + \overline{r}_1 \times \overline{F}_{1n} = 0 \\ &\overline{r}_2 \times \overline{F}_2 + \overline{r}_2 \times \overline{F}_{21} + \ldots + \overline{r}_2 \times \overline{F}_{2n} = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\overline{r}_n \times \overline{F}_n + \overline{r}_n \times \overline{F}_{n1} + \ldots + \overline{r}_n \times \overline{F}_{nn} = 0 \end{aligned}$$

6.1.2. Teorema solidificării

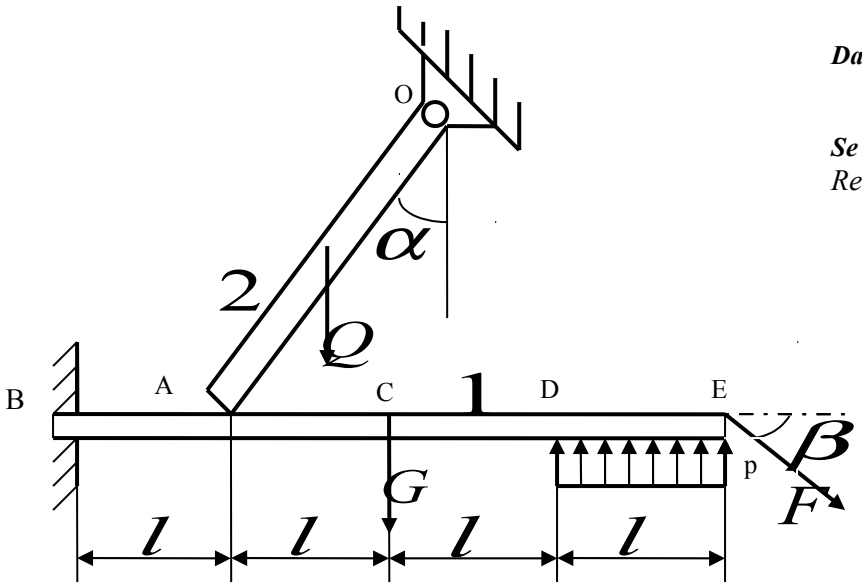
$$\sum_{i=1}^{n} \overline{F}_i = 0$$

$$\sum_{i=1}^{n} \overline{r}_i \times \overline{F}_i = 0$$

6.2. Echilibrul sistemelor fără frecare

6.2.1. Probleme rezolvate

1)



Date:

Se cer:
Reacțiuni în O, A, B.

Fig. 6.1

Rezolvare:

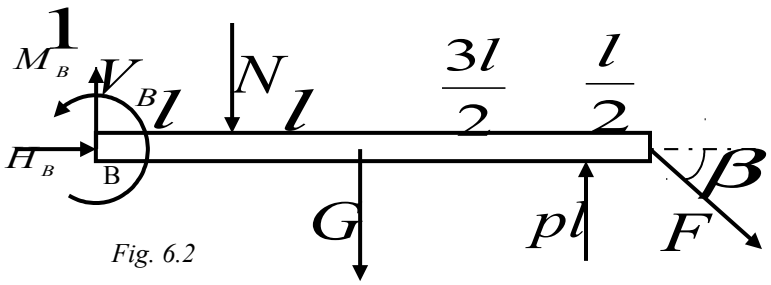
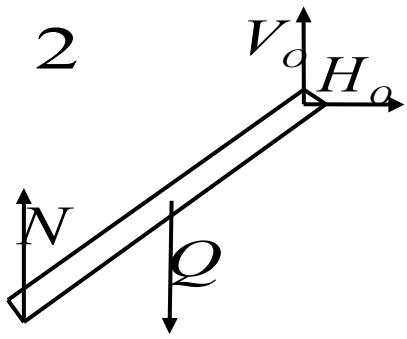


Fig. 6.2



Bara 1:

$$\sum F_{ix} = 0 : F\cos\beta - H_B = 0$$
$$\sum F_{iy} = 0 : V_B - N - G + pl - F\sin\beta = 0$$
$$\sum M_{iB} = 0 : M_B - N\cdot l - G\cdot 2l + pl\cdot\frac{7l}{2} - F\sin\beta\cdot 4l = 0$$

Bara 2:

$$\sum F_{ix} = 0 : H_O = 0$$
$$\sum F_{iy} = 0 : V_O - Q + N = 0$$
$$\sum M_{iO} = 0 : Q \cdot l \sin\alpha - N \cdot 2l \sin\alpha = 0$$

Rezultă

$$H_B = F\cos\beta; V_B = \frac{Q}{2} + G - pl + F\sin\beta; M_B = \frac{Q}{2}l + 2Gl - \frac{7}{2}pl^2 + F\sin\beta \cdot 4l;$$
$$H_O = 0; V_O = \frac{Q}{2}; N = \frac{Q}{2}.$$

2)

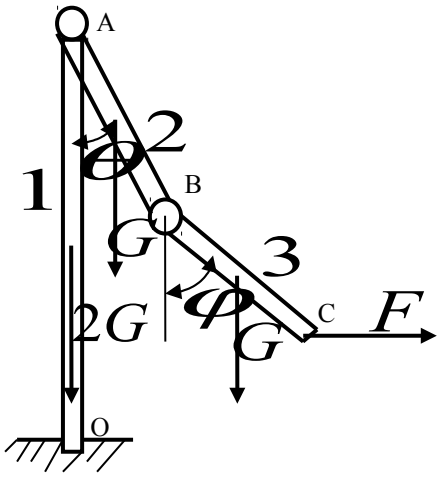


Fig. 6.3

Date:

Se cer:
pentru echilibru;
Reacţiuni în O, A şi B.

Rezolvare:

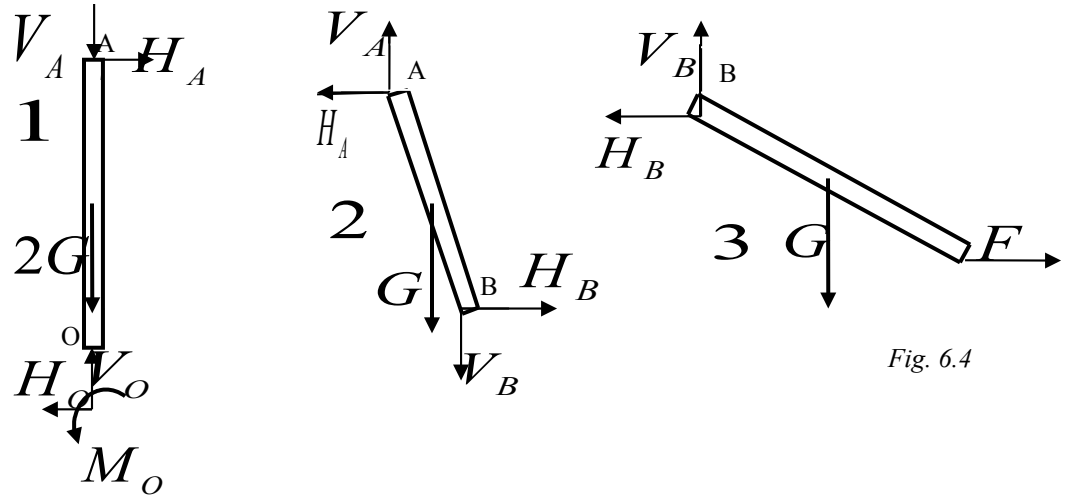


Fig. 6.4

Bara 1:

$$\sum F_{ix} = 0 : H_A - H_O = 0$$
$$\sum F_{iy} = 0 : V_O - 2G - V_A = 0$$
$$\sum M_{iO} = 0 : H_A \cdot 2l - M_O = 0$$

Bara 2:

$$\sum F_{ix} = 0 : H_B - H_A = 0$$
$$\sum F_{iy} = 0 : V_A - G - V_B = 0$$
$$\sum M_{iA} = 0 : H_B l \cos\theta - V_B l \sin\theta - G\frac{l}{2}\sin\theta = 0$$

Bara 3:

$$\sum F_{ix} = 0 : F - H_B = 0$$
$$\sum F_{iy} = 0 : V_B - G = 0$$
$$\sum M_{iB} = 0 : Fl\cos\varphi - G\frac{l}{2}\sin\varphi = 0$$

Rezultă:

$$tg\theta = \frac{2F}{3G} \Rightarrow \theta = arctg\frac{2F}{3G}; tg\varphi = \frac{2F}{G} \Rightarrow \varphi = arctg\frac{2F}{G};$$
$$H_O = H_A = H_B = F; V_O = 4G; V_A = 2G; V_B = G; M_O = 2Fl.$$

6.2.2. Probleme propuse

1)

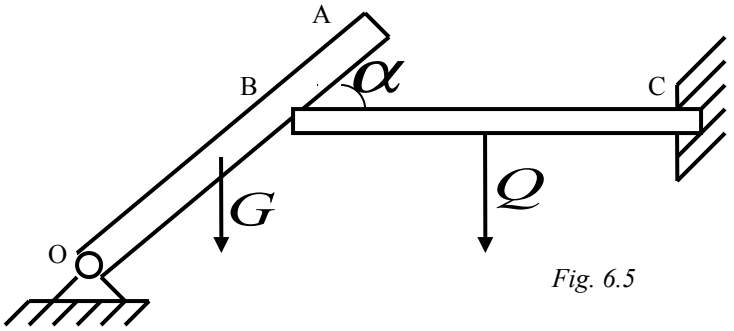


Fig. 6.5

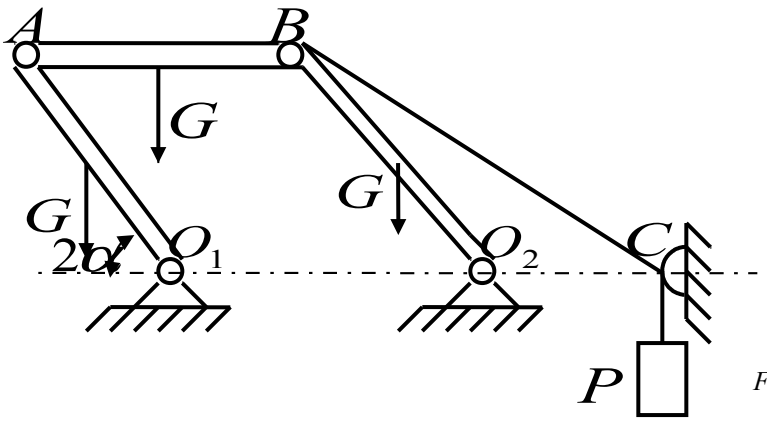
Date:

Se cer:
Reacțiunile din O; B; C.

Răspuns:

$$H_O = \frac{G}{3}\sin 2\alpha; V_O = G - \frac{2}{3}G\cos^2\alpha; N = \frac{2}{3}G\cos\alpha;$$
$$H_C = -\frac{G}{3}\sin 2\alpha; V_C = Q + \frac{2}{3}G\cos^2\alpha; M_C = Q\cdot a + \frac{4}{3}G\cdot a\cos^2\alpha.$$

2)



Date:

Se cer:
Reacţiunile din:

Fig.6.6

Răspuns:

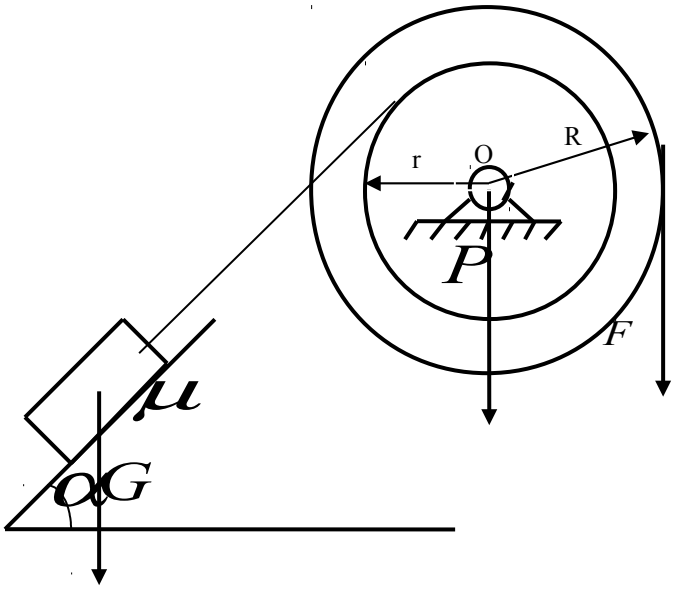
$$P = 2G\frac{\cos 2\alpha}{\sin \alpha}; H_{O1} = H_A = H_B = G \cdot ctg2\alpha; V_{O1} = \frac{3}{2}G; V_A = V_B = \frac{G}{2};$$

$$H_{O2} = G \cdot ctg2\alpha(1 - 2 \cdot ctg\alpha \cdot \sin 2\alpha); V_{O2} = \frac{G}{2}(3 + 4\cos 2\alpha).$$

6.3. Echilibrul sistemelor cu frecare

6.3.1. Probleme rezolvate

1)



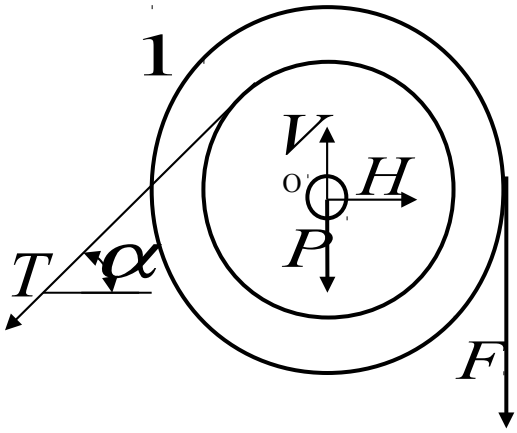
Date:

Se cer:

pentru echilibru.

Fig. 6.7

Rezolvare:



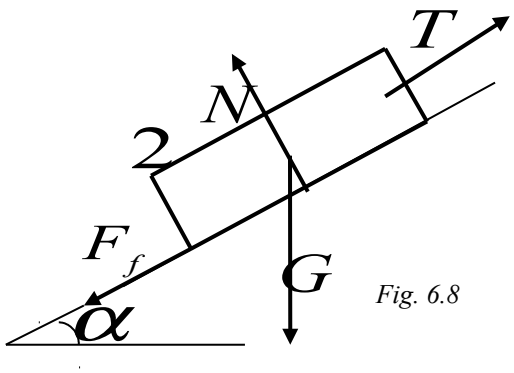


Fig. 6.8

Corpul 1:

$$\sum F_{ix} = 0 : H - T\cos\alpha = 0$$
$$\sum F_{iy} = 0 : V - P - T\sin\alpha = 0$$
$$\sum M_{iO} = 0 : FR - Tr = 0$$

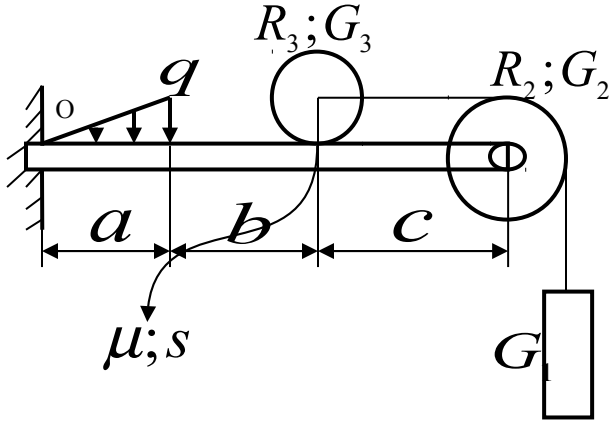
Corpul 2:

$$\sum F_{ix} = 0 : T - G\sin\alpha - F_f = 0$$
$$\sum F_{iy} = 0 : N - G\cdot\cos\alpha$$
$$F_f \le \mu \cdot N$$

Rezultă:

$$F \le \frac{r}{R} G(\sin\alpha + \mu\cdot\cos\alpha); T = F\frac{R}{r}; H = F\frac{R}{r}\cdot\cos\alpha; V = P + F\frac{R}{r}\cdot\sin\alpha; N = G\cos\alpha.$$

2)



Date:

Se cer:
, pentru echilibru; reacțiuni.

Fig. 6.9

Rezolvare:

T_2

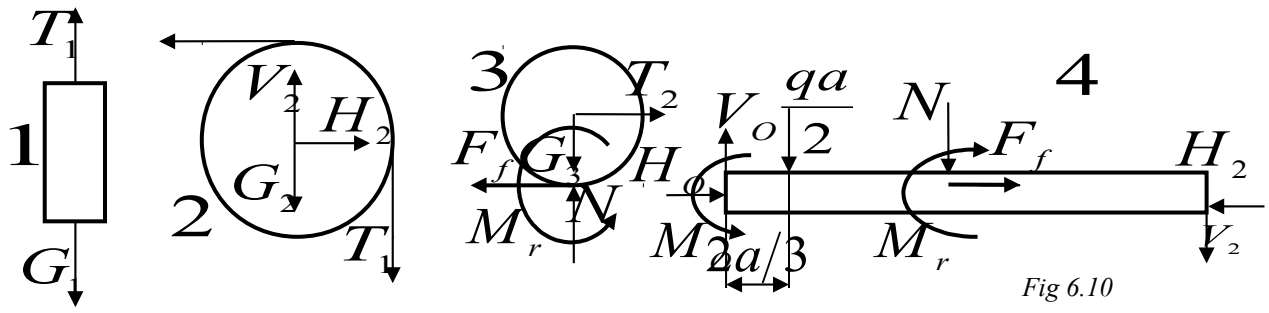


Fig 6.10

Corpul 1:

$$\sum F_{iy} = 0 : G_1 - T_1 = 0 \Rightarrow G_1 = T_1;$$

Corpul 2:

$$\sum F_{ix} = 0 : H_2 - T_2 = 0$$
$$\sum F_{iy} = 0 : V_2 - G_2 - T_1 = 0$$
$$\sum M_{iO2} = 0 : T_1 R_2 - T_2 R_2 = 0$$

Corpul 3:

$$\sum F_{ix} = 0 : T_2 - F_f = 0$$
$$\sum F_{iy} = 0 : N - G_3 = 0$$
$$\sum M_{iO3} = 0 : F_f R_3 - M_r = 0$$
$$F_f \le \mu \cdot N; M_r \le s \cdot N$$

Corpul 4:

$$\sum F_{ix} = 0 : H_O + F_f - H_2 = 0$$
$$\sum F_{iy} = 0 : V_O - \frac{qa}{2} - N - V_2 = 0$$
$$\sum M_{iO} = 0 : M_O - \frac{qa}{2} \cdot \frac{2a}{3} - N(a+b) - V_2(a+b+c) = 0$$

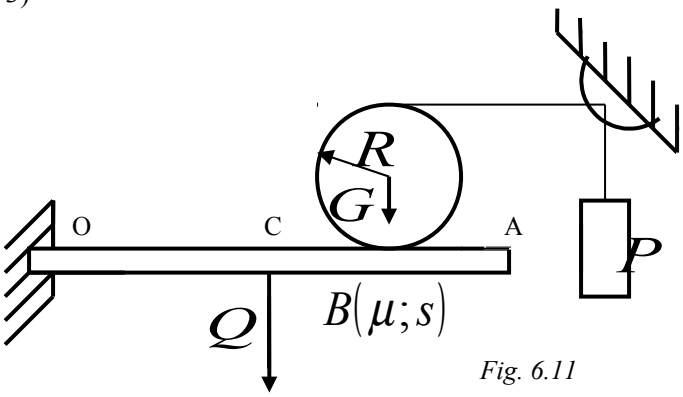
Rezultă:

$$G_1 \le G_1^{alunecare} = \mu \cdot G_3; G_1 \le G_1^{rostogolire} = \frac{s}{R_3} \cdot G_3; \Rightarrow G_1 \le \min\left(\mu \cdot G_3; \frac{s}{R_3} G_3 \right)$$

$$T_1 = T_2 = G_1; H_2 = T_2 = G_1; V_2 = G_1 + G_2; N = G_3; H_O = 0; V_O = \frac{qa}{2} + G_1 + G_2 + G_3;$$

$$M_O = \frac{qa^2}{3} + G_3(a+b) + (G_1 + G_1) \cdot (a+b+c).$$

3)



Date:

Se cer:
pentru echilibru;
Cum începe mişcarea dacă
Reacţiuni din O şi B pentru

Fig. 6.11

Rezolvare:

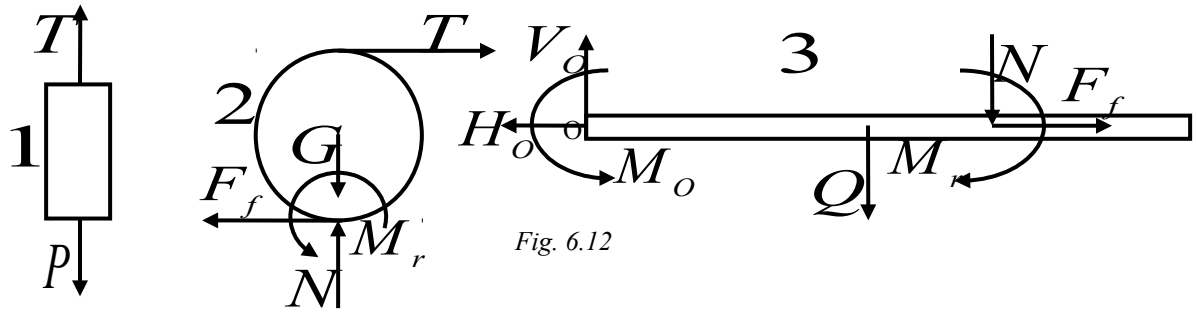


Fig. 6.12

Corpul 1:

$$\sum F_{iy} = 0 : P - T = 0 \Rightarrow P = T;$$

Corpul 2:

$$\sum F_{ix} = 0 : T - F_f = 0$$
$$\sum F_{iy} = 0 : N - G = 0$$
$$\sum M_{iB} = 0 : T \cdot 2R - M_r = 0$$
$$F_f \le \mu \cdot N; M_r \le s \cdot N$$

Corpul 3:

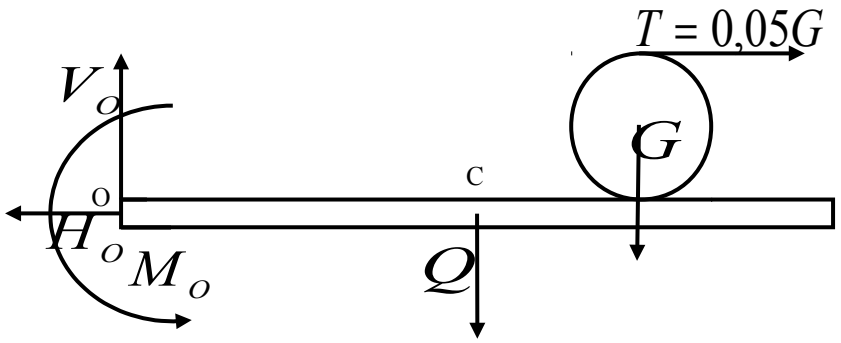
$$\sum F_{ix} = 0 : F_f - H_O = 0$$
$$\sum F_{iy} = 0 : V_O - Q - N = 0$$
$$\sum M_{iO} = 0 : M_O - Q \cdot \frac{l}{2} - N\frac{3l}{4} - M_r = 0$$

Rezultă:

$$alunecare : P \le 0{,}1G; rostogolire : P \le 0{,}05G \Rightarrow P_{\max} = 0{,}05G \Rightarrow N = G;$$
$$H_O = 0{,}05G; V_O = G + Q; M_O = Q\frac{l}{2} + G\frac{3}{4}l + 0{,}1GR.$$

Metoda solidificării:

Se foloseşte pentru determinarea reacțiunilor din încastrarea O:

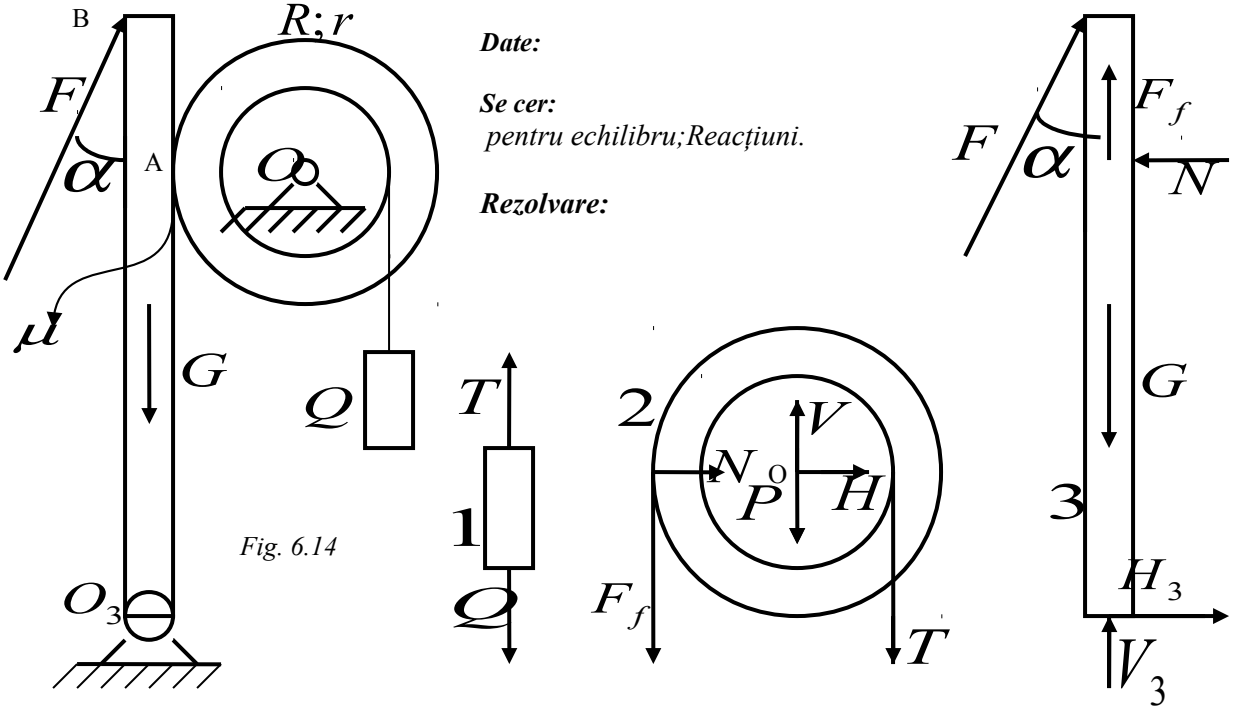


$$\sum F_{ix} = 0 :$$
$$\sum F_{iy} = 0 :$$
$$\sum M_{iO} = 0 :$$

Fig. 6.13

$$T - H_O = 0 \qquad H_O = T = P = 0{,}05G;$$
$$V_O - Q - G = 0 \qquad V_O = G + Q;$$
$$M_O - Q\frac{l}{2} - G\frac{3}{4}l - 0{,}1GR = 0 \qquad M_O = \frac{Ql}{2} + \frac{3Gl}{4} + 0{,}1GR.$$

4)



Date:

Se cer:
pentru echilibru;Reacţiuni.

Rezolvare:

Fig. 6.14

Corpul 1:

$$\sum F_{iy} = 0 : Q - T = 0 \Rightarrow T = Q$$

Corpul 2:

$$\sum F_{ix} = 0 : H + N = 0$$
$$\sum F_{iy} = 0 : V - F_f - T - P = 0$$
$$\sum M_{iO} = 0 : Tr - F_f R = 0$$
$$F_f \le \mu \cdot N$$

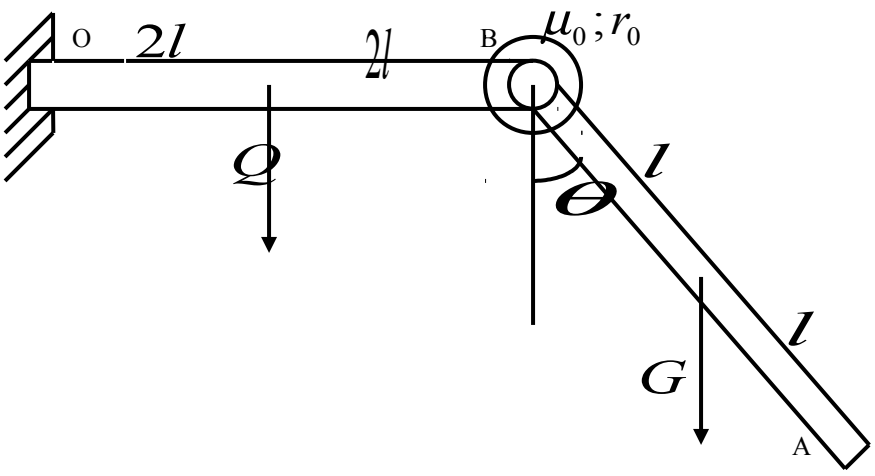
Corpul 3:

$$\sum F_{ix} = 0 : H_3 + F \sin\alpha - N = 0$$
$$\sum F_{iy} = 0 : V_3 + F_f + F \cos\alpha - G = 0$$
$$\sum M_{iO3} = 0 : F \sin\alpha \cdot 3l - N \cdot 2l = 0$$

Rezultă:

$$F_f = \frac{Tr}{R} = \frac{Qr}{R}; N = \frac{3}{2} F \sin\alpha \Rightarrow \frac{Qr}{R} \le \mu \cdot \frac{3}{2} F \sin\alpha \Rightarrow F \ge \frac{2}{3} \frac{Qr}{\mu \cdot R \cdot \sin\alpha}; H = -N = -\frac{3}{2} F \sin\alpha;$$
$$V = F_f + T + P = P + \frac{Q}{R}(R + r); H_3 = \frac{3}{2} F \sin\alpha - F \sin\alpha = \frac{F}{2} \sin\alpha; V_3 = G - F \cos\alpha - \frac{Qr}{R}.$$

5)



Date:

Se cere:
pentru echilibru.

Fig. 6.15

Rezolvare:

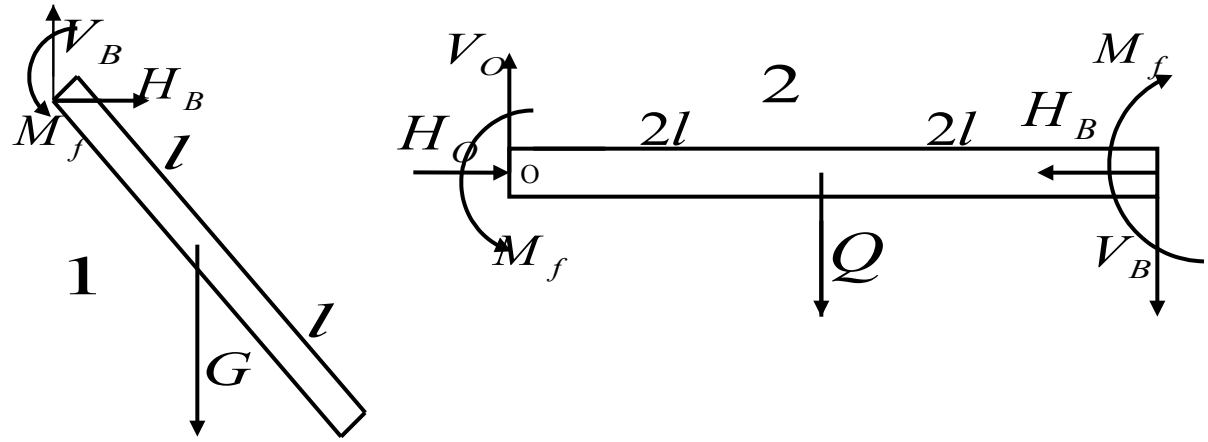


Fig. 6.16

Corpul 1:

$$\sum F_{ix} = 0 : H_B = 0$$
$$\sum F_{iy} = 0 : V_B - G = 0$$
$$\sum M_{iB} = 0 : M_f - Gl\sin\theta = 0$$
$$M_f \le \mu_0 r_0 \sqrt{H_B^2 + V_B^2}$$

Corpul 2:

$$\sum F_{ix} = 0 : H_O - H_B = 0$$
$$\sum F_{iy} = 0 : V_O - Q - V_B = 0$$
$$\sum M_{iO} = 0 : M_O - Q \cdot 2l - V_B \cdot 4l - M_f = 0$$

Rezultă:

$$M_f = Gl \sin\theta \Rightarrow Gl \sin\theta \le \mu_0 r_0 G \Rightarrow \sin\theta \le \mu_0 \frac{r_0}{l}$$

6)

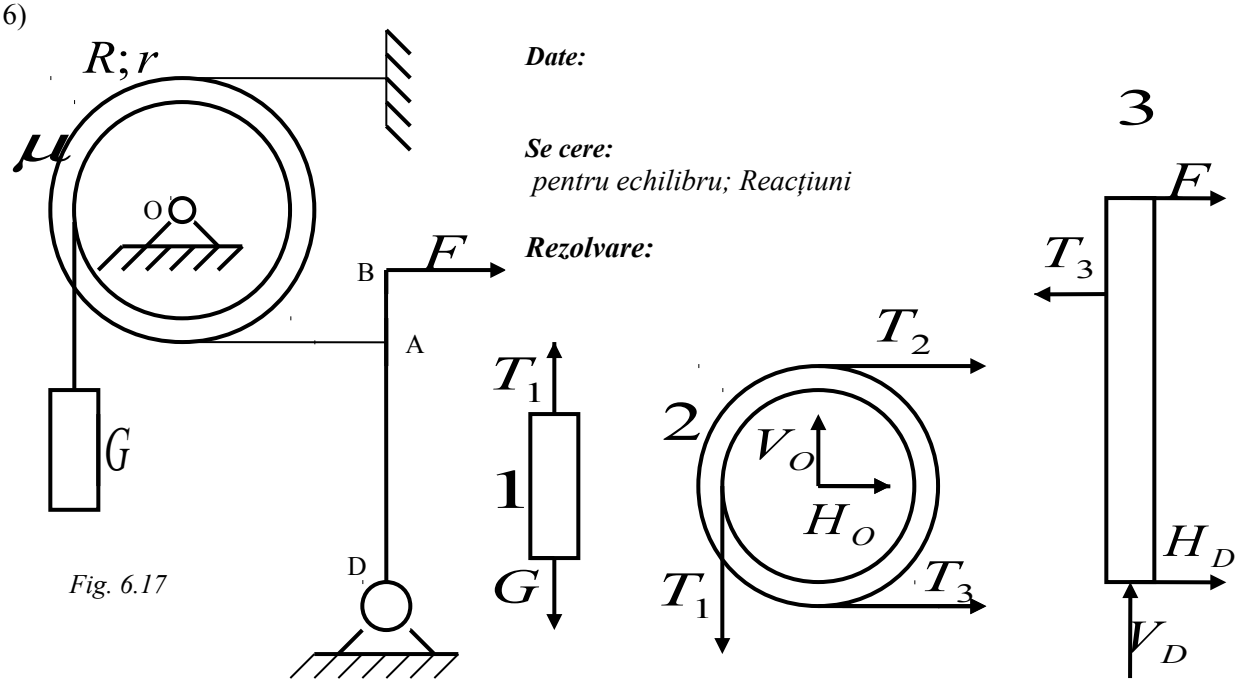


Fig. 6.17

Date:

Se cere:
pentru echilibru; Reacțiuni

Rezolvare:

Corpul 1:

$$\sum F_{iy} = 0 : G - T_1 = 0 \Rightarrow T_1 = G$$

Corpul 2:

$$\sum F_{ix} = 0 : H_O + T_2 + T_3 = 0$$
$$\sum F_{iy} = 0 : V_O - T_1 = 0$$
$$\sum M_{iO} = 0 : T_1 r - T_2 R + T_3 R = 0$$
$$T_2 \le T_3 e^{\mu \cdot \pi}$$

Corpul 3:

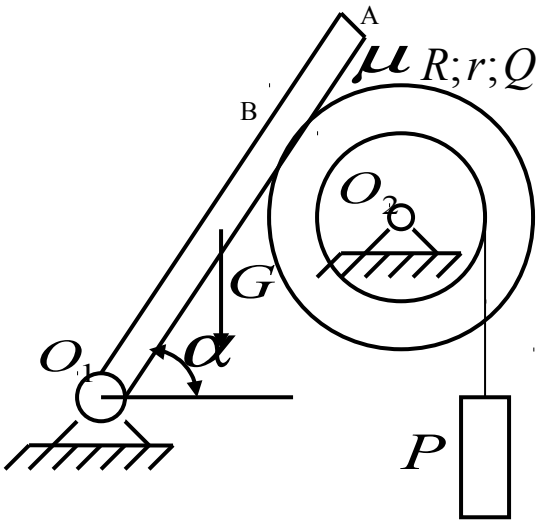
$$\sum F_{ix} = 0 : F + H_D - T_3 = 0$$
$$\sum F_{iy} = 0 : V_D = 0$$
$$\sum M_{iD} = 0 : T_3 \cdot 3l - F \cdot 4l = 0$$

Rezultă:

$$T_3 = \frac{4}{3}F; T_2 = G\frac{r}{R} + \frac{4}{3}F \Rightarrow G\frac{r}{R} + \frac{4}{3}F \le \frac{4}{3}Fe^{\mu\cdot\pi} \Rightarrow \frac{4}{3}F\left(e^{\mu\cdot\pi} - 1\right) \ge G\frac{r}{R} \Rightarrow F \ge \frac{3}{4}\cdot\frac{r}{R}\cdot\frac{G}{e^{\mu\cdot\pi} - 1}$$
$$\Rightarrow F_{\min} = \frac{3}{4}\cdot\frac{r}{R}\cdot\frac{G}{e^{\mu\cdot\pi} - 1}; H_O = -G\frac{r}{R} - \frac{8}{3}F; V_O = G; H_D = \frac{F}{3}; V_D = 0.$$

6.3.2. Probleme propuse

1)



Date:

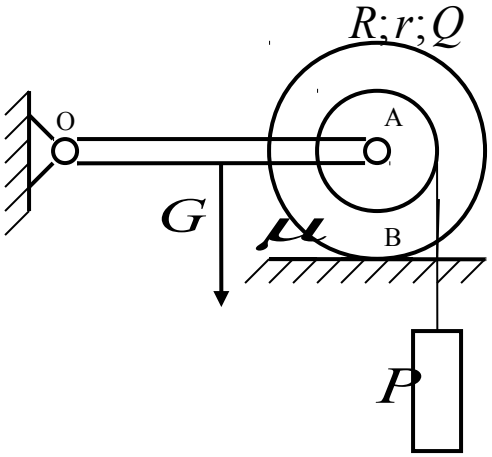
Se cer:
pentru echilibru; Reacțiunile din

Fig. 6.18

Răspuns:

$$P \le P_{\max} = \frac{2}{3}\frac{R}{r}\mu \cdot G\cos\alpha; H_{O1} = \frac{2}{3}G\cos\alpha(\sin\alpha - \mu\cdot\cos\alpha); V_{O1} = G - \frac{2}{3}G\cos\alpha(\cos\alpha + \mu\cdot\sin\alpha);$$
$$N = \frac{2}{3}G\cos\alpha; H_{O2} = -\frac{2}{3}G\cos\alpha(\sin\alpha - \mu\cdot\cos\alpha); V_{O2} = P_{\max} + Q + \frac{2}{3}G\cos\alpha(\cos\alpha + \mu\cdot\sin\alpha).$$

2)



Date:

Se cer:

pentru echilibru; Reacţiunile din O pentru

Fig. 6.19

Răspuns:

$$P \le P_{\max} = \frac{\mu \cdot (G + 2 \cdot Q) R}{2(r - \mu \cdot R)};$$

Dacă numitorul este pozitiv sau nul atunci sistemul se autoblochează.

$$H_O = \mu \cdot \left(\frac{G}{2} + Q + P_{\max} \right); V_O = \frac{G}{2}.$$

3)

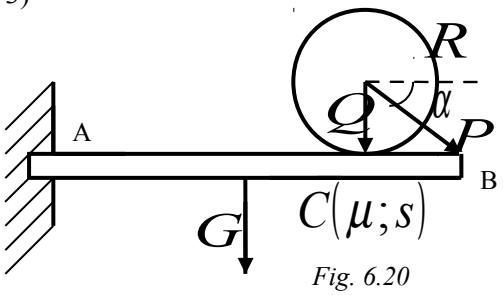


Fig. 6.20

Date:

Se cer:

pentru echilibru; reacţiunile din A cu metoda izolării corpurilor şi cu metoda solidificării.

Răspuns:

$$P \le \min\left[\frac{\mu \cdot G}{\cos\alpha - \mu \cdot \sin\alpha}; \frac{s \cdot G}{R\cos\alpha - s\sin\alpha} \right]; H_A = P\cos\alpha; V_A = G + Q + P\sin\alpha;$$

$$M_A = Gl + \frac{4}{3}Ql + PR\cos\alpha + \frac{4}{3}Pl\sin\alpha.$$

4)

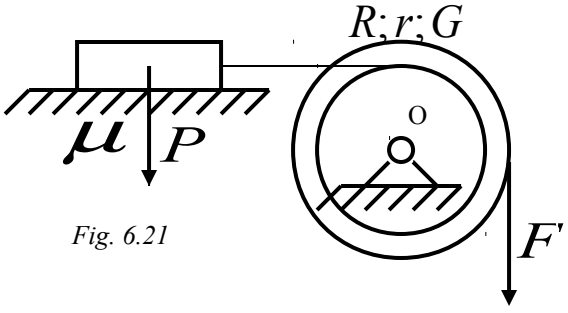


Fig. 6.21

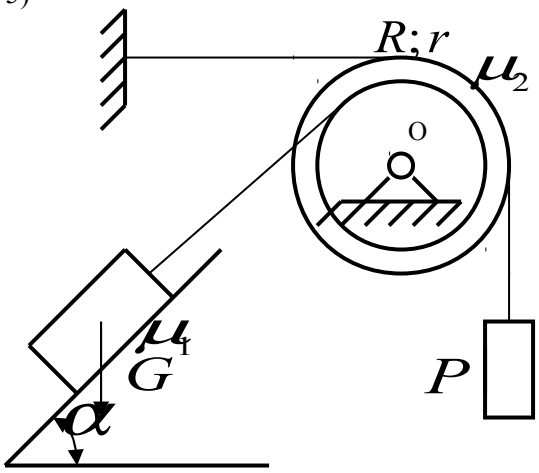
Date:

Se cer:

pentru echilibru.

Răspuns:

5)



Date:

Se cere:
pentru echilibru.

Răspuns:

Fig. 6.22

6.4. Sisteme de bare articulate (Grinzi cu zăbrele)

6.4.1. Generalități

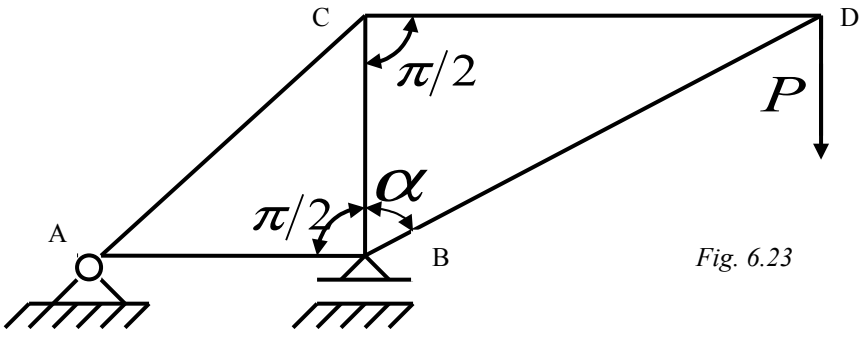
Corectitudinea unui astfel de sistem se verifică, cu relația:

$$b = 2n - 3$$

unde: b – numărul barelor sistemului;
n – numărul nodurilor sistemului

6.4.2. Probleme rezolvate

1)

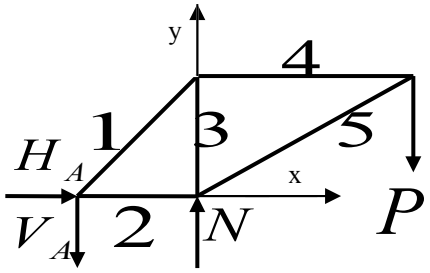


Date:

Se cer:
Tensiunile din bare.

Fig. 6.23

Rezolvare:



verificare : $b = 5; n = 4 \Rightarrow 5 = 2 \cdot 4 - 3.$

$\sum F_{ix} = 0 : H_A = 0$

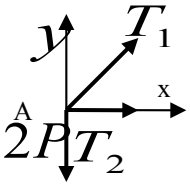
$\sum F_{iy} = 0 : N - V_A - P = 0$

$\sum M_{iA} = 0 : N \cdot l - 3Pl = 0$

$H_A = 0;$

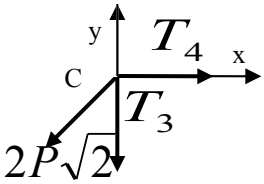
$N = 3P;$

$V_A = 2P.$



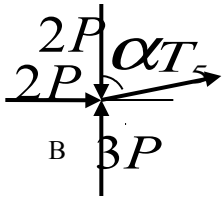
$T_1 \frac{\sqrt{2}}{2} + T_2 = 0 \Rightarrow T_2 = -2P$

$T_1 \frac{\sqrt{2}}{2} - 2P = 0 \Rightarrow T_1 = 2P\sqrt{2}$



$T_4 - 2P = 0 \Rightarrow T_4 = 2P$

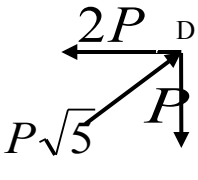
$-2P - T_3 = 0 \Rightarrow T_3 = -2P$



$\sin\alpha = \frac{2}{\sqrt{5}}; \cos\alpha = \frac{1}{\sqrt{5}}$

$3P - 2P + T_5 \frac{1}{\sqrt{5}} = 0 \Rightarrow T_5 = -P\sqrt{5}$

$3P - 2P - P = 0$



$P\sqrt{5} \cdot \frac{2}{\sqrt{5}} - 2P = 0$

$P\sqrt{5} \frac{1}{\sqrt{5}} - P = 0$

2)

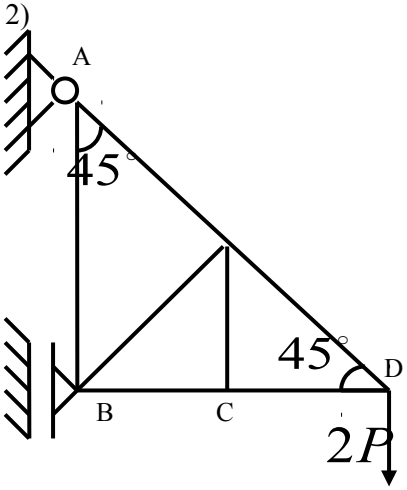
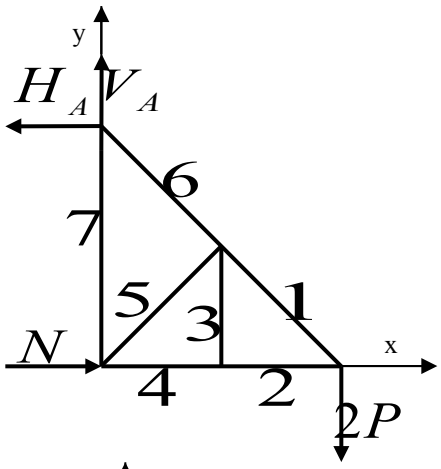


Fig. 6.24

Date:

Se cer:
Tensiunile din bare.

Rezolvare:

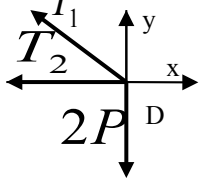


$$verificare: b = 7; n = 5 \Rightarrow 7 = 2 \cdot 5 - 3.$$

$$\sum F_{ix} = 0 : N - H_A = 0 \Rightarrow H_A = 2P$$

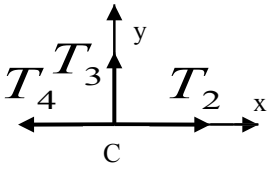
$$\sum F_{iy} = 0 : V_A - 2P = 0 \Rightarrow V_A = 2P$$

$$\sum M_{iA} = 0 : N \cdot 2l - 2P \cdot 2l = 0 \Rightarrow N = 2P$$



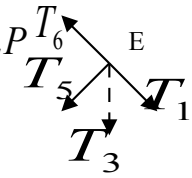
$$-T_1 \frac{\sqrt{2}}{2} - T_2 = 0 \Rightarrow T_2 = -2P$$

$$T_1 \frac{\sqrt{2}}{2} - 2P = 0 \Rightarrow T_1 = 2P\sqrt{2}$$



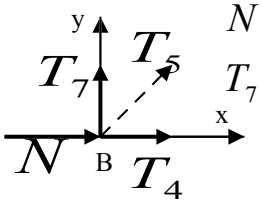
$$T_2 - T_4 = 0 \Rightarrow T_4 = -2P$$

$$T_3 = 0$$



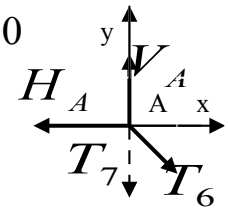
$$T_1 - T_6 = 0 \Rightarrow T_6 = 2P\sqrt{2}$$

$$T_5 = 0$$



$$N + T_4 = 0$$

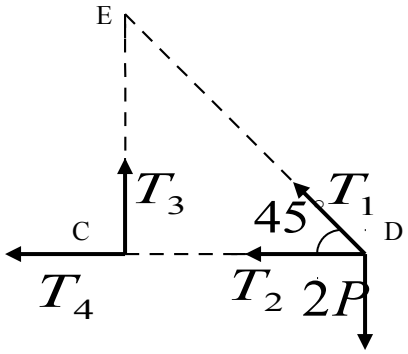
$$T_7 = 0$$



$$T_6 \frac{\sqrt{2}}{2} - H_A = 0 \Rightarrow 2P\sqrt{2} \cdot \frac{1}{\sqrt{2}} - 2P = 0$$

$$V_A - T_6 \frac{\sqrt{2}}{2} = 0 \Rightarrow 2P - 2P\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 0$$

Verificare cu metoda secțiunilor pentru barele 1; 3 şi 4:



$$\sum M_D = 0 : T_3 l = 0 \Rightarrow T_3 = 0$$

$$\sum M_E = 0 : -T_4 l - 2Pl = 0 \Rightarrow T_4 = -2P$$

$$\sum M_C = 0 : T_1 \frac{\sqrt{2}}{2} l - 2Pl = 0 \Rightarrow T_1 = 2P\sqrt{2}$$

3)

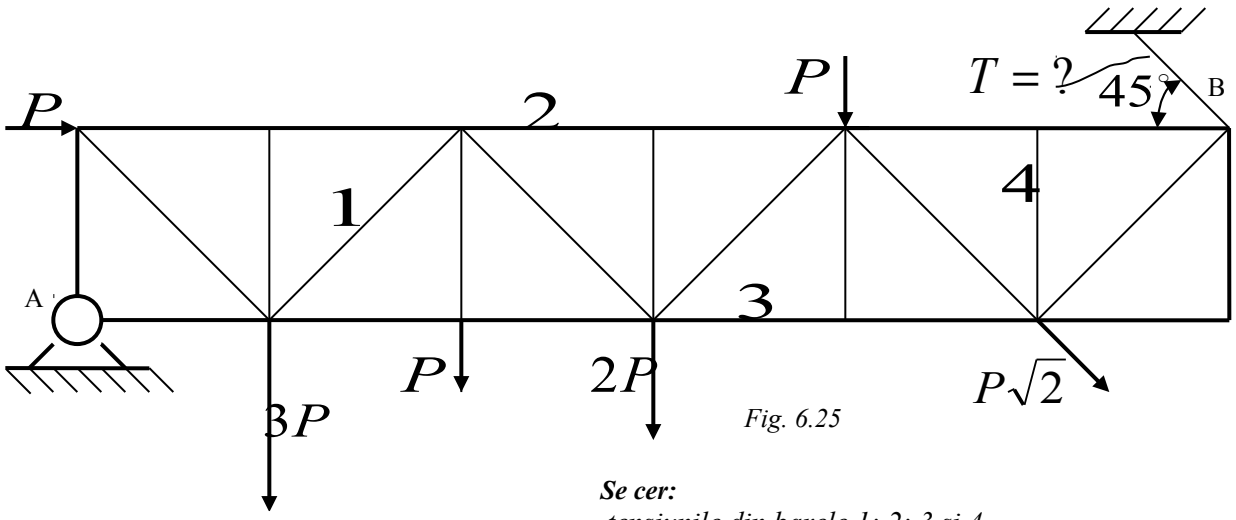
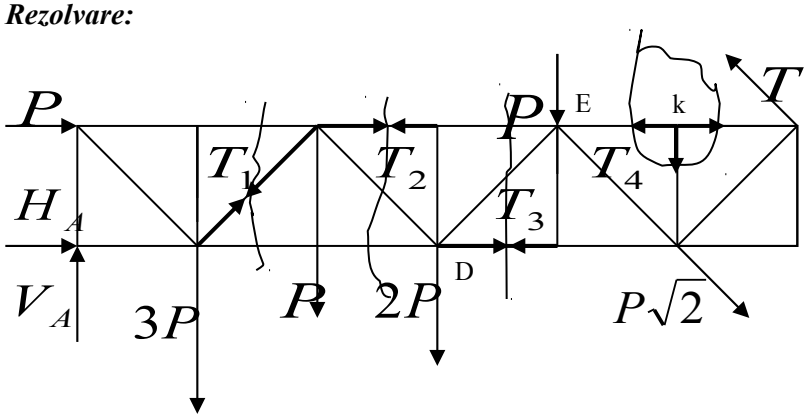


Fig. 6.25

Se cer:
tensiunile din barele 1; 2; 3 şi 4.

Date:
Barele perpendiculare au lungimea l; P

Rezolvare:



$$H_A + 2P - T\frac{\sqrt{2}}{2} = 0 \Rightarrow H_A = P$$

$$V_A - 8P + T\frac{\sqrt{2}}{2} = 0 \Rightarrow V_A = 5P$$

$$T\frac{\sqrt{2}}{2}\cdot 7l - Pl - 3Pl - P\cdot 2l - 2P\cdot 3l - P\cdot 4l - P\cdot 5l = 0 \Rightarrow \frac{7}{2}T\sqrt{2} = 21P \Rightarrow T\sqrt{2} = 6P \Rightarrow T = 3P\sqrt{2}$$

Pentru porțiunea din partea de grindă situată în stânga secțiunii *1* se aplică relația:

$$\sum F_{iy} = 0 : 5P - 3P + T_1\frac{\sqrt{2}}{2} = 0 \Rightarrow T_1 = -2P\sqrt{2}$$

Pentru determinarea tensiunii în secțiunea *2* a grinzii se aplică, în partea stângă a secțiunii, relația:

$$\sum M_{iD} = 0 : -Pl - 5P\cdot 3l - 3P\cdot 2l + Pl - T_2 l = 0 \Rightarrow T_2 = -21P$$

Pentru determinarea tensiunii în secțiunea *3* a grinzii se aplică, în partea dreaptă a secțiunii, relația:

$$\sum M_{iE} = 0 : T_3 l - 3P\sqrt{2}\cdot 2l\frac{\sqrt{2}}{2} \Rightarrow T_3 = 6P$$

Pentru determinarea tensiunii în nodul *4* se aplică relația:

$$\sum F_{iy} = 0 \Rightarrow T_4 = 0$$

6.4.3. Probleme propuse

1)

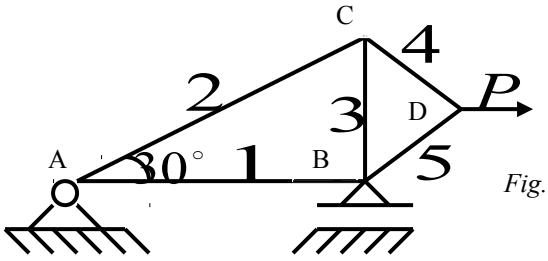


Fig. 6.26

Date:

Se cer:
Tensiunile din bare.

Răspuns:

$$T_1 = \frac{P}{2}; T_2 = \frac{P}{\sqrt{3}}; T_3 = -\frac{P}{\sqrt{3}}; T_4 = \frac{P}{\sqrt{3}}; T_5 = \frac{P}{\sqrt{3}}.$$

2)

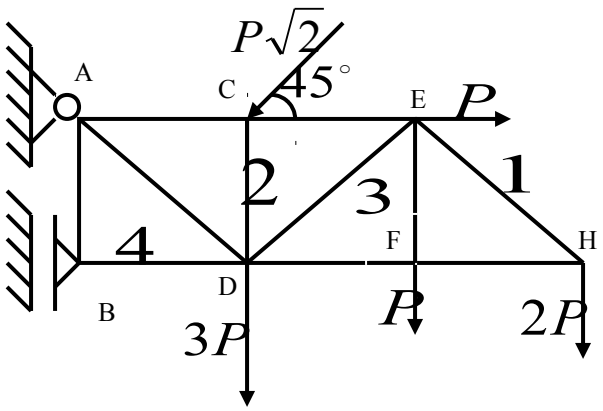


Fig. 6.27

Date:

Se cer:
Reacțiunile din A şi B; Tensiunile din barele 1, 2, 3 şi 4.

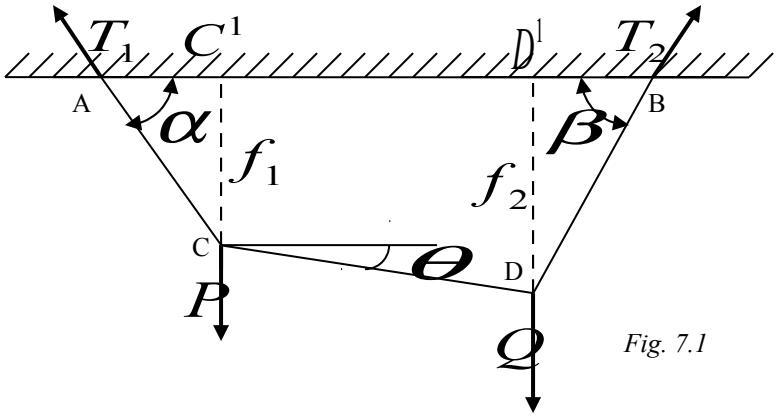
Răspuns:

$$H_A = -12P; V_A = 7P; N_B = 12P; T_1 = 2P\sqrt{2}; T_2 = -P; T_3 = -3P\sqrt{2}; T_4 = -12P.$$

CAPITOLUL 7

STATICA FIRELOR

7.1. Probleme rezolvate

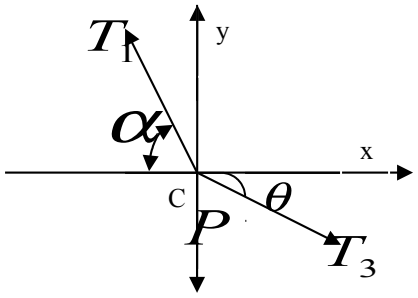


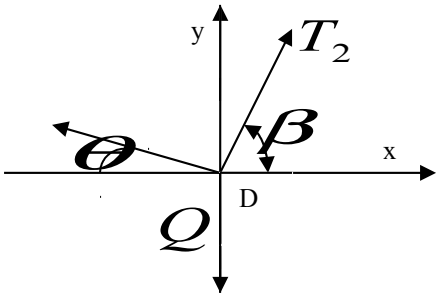
Date:

Se cer:

Fig. 7.1

Rezolvare:





Întrucât firul nu are greutate, pe porțiunile *AC*, *CD* şi *DB* tensiunile sunt constante.

$$C:\begin{cases}\sum F_{ix}=0:-T_1\cos\alpha+T_3\cos\beta=0\\ \sum F_{iy}=0:T_1\sin\alpha-T_3\sin\theta-P=0\end{cases}$$

$$D:\begin{cases}\sum F_{ix}=0:-T_3\cos\theta+T_2\cos\beta=0\\ \sum F_{iy}=0:T_3\sin\theta-T_2\sin\beta-Q=0\end{cases}$$

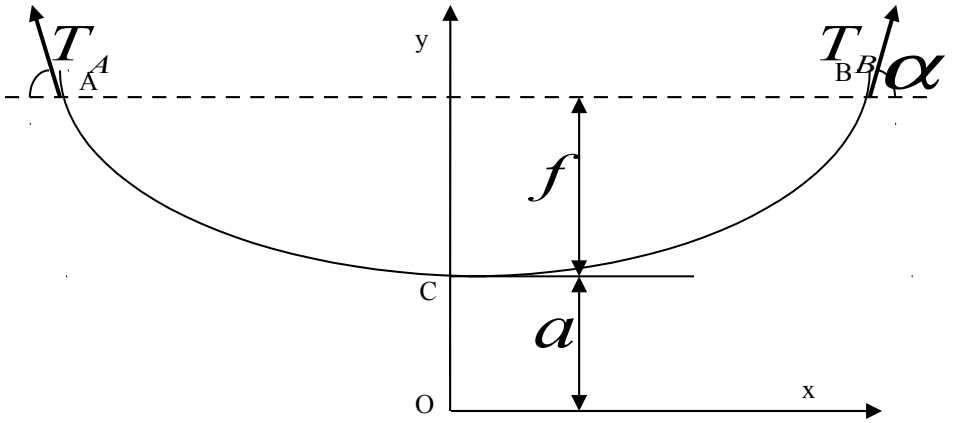
Rezultă:

$$T_1=\frac{(P+Q)\cos\beta}{\sin(\alpha+\beta)};T_2=\frac{(P+Q)\cos\alpha}{\sin(\alpha+\beta)};T_3=\sqrt{(T_1\cos\alpha)^2+(T_2\sin\beta-Q)^2};tg\theta=\frac{Q\cdot tg\alpha-P\cdot tg\beta}{Q+P};$$

$$l=\frac{f_1}{\sin\alpha}+\frac{f_2-f_1}{\sin\theta}+\frac{f_2}{\sin\beta};AB=f_1\cdot tg\alpha+(f_2-f_1)\cdot tg\theta+f_2\cdot tg\beta.$$

2)

α



Date:

Se cer:

Fig. 7.2

Rezolvare:

$$y = a \cdot ch\frac{x}{a}; T = p \cdot y; H = p \cdot a; AC = CB = a \cdot sh\frac{x_B}{a} \Rightarrow T_B = p \cdot y_B; T_{\min} = H = p \cdot a = T_B \cos\alpha;$$

$$L = a \cdot sh\frac{x_B}{a} \; deoarece: tg\alpha = y' \Big| x = x_B \Rightarrow L = a \cdot sh\frac{x}{a}\Big| x = x_B = a \cdot sh\frac{x}{a}$$

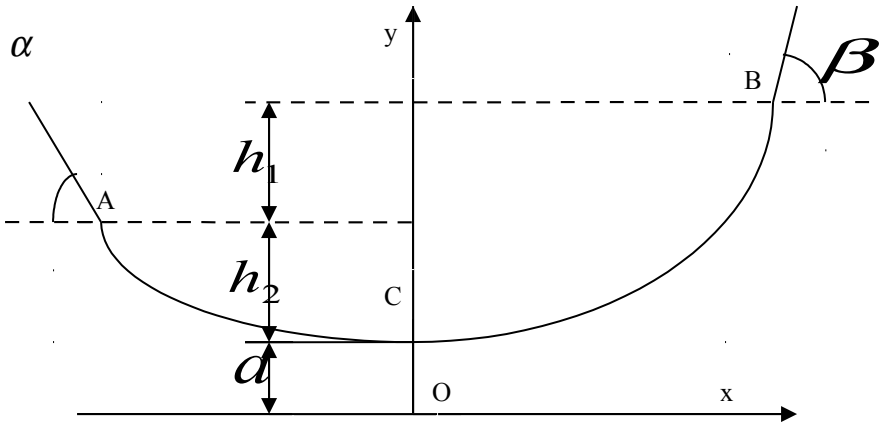
$$L = a \cdot tg\alpha \Rightarrow a = \frac{L}{tg\alpha} \Rightarrow H = \frac{pL}{tg\alpha}$$

$$T_A = T_B = \frac{H}{\cos\alpha} = \frac{pL\cos\alpha}{\sin\alpha\cos\alpha} = \frac{pL}{\sin\alpha}$$

$$a + f = y_B = \frac{T_B}{p} = \frac{pL}{p\sin\alpha} = \frac{L}{\sin\alpha} \Rightarrow f = \frac{L}{\sin\alpha} - a = \frac{L}{\sin\alpha} - \frac{L}{tg\alpha} = \frac{L \cdot (1 - \cos\alpha)}{\sin\alpha}$$

7.2. Probleme propuse

1)



Date:

Se cer:

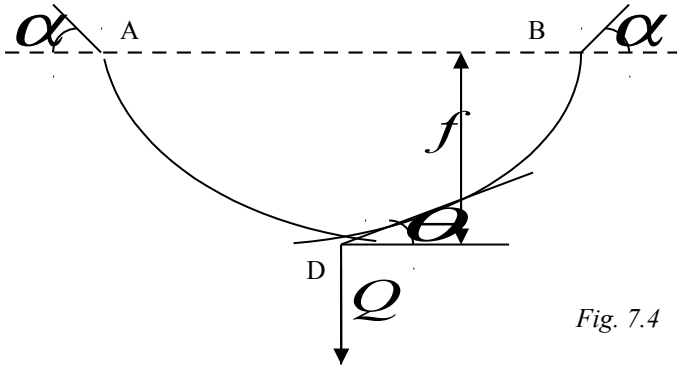
Fig. 7.3

Răspuns:

$$T_A = \frac{p \cdot l \cos\beta}{\sin(\alpha+\beta)}; T_B = \frac{p \cdot l \cos\alpha}{\sin(\alpha+\beta)}; T_C = \frac{p \cdot l \cos\alpha \cos\beta}{\sin(\alpha+\beta)}; l_{AC} = \frac{L \sin\alpha \cos\beta}{\sin(\alpha+\beta)};$$

$$l_{CB} = \frac{L \cos\alpha \sin\beta}{\sin(\alpha+\beta)}; h_1 = \frac{L \cdot (\cos\alpha - \cos\beta)}{\sin(\alpha+\beta)}; h_2 = \frac{L \cos\beta \cdot (1-\cos\alpha)}{\sin(\alpha+\beta)}.$$

2)



Date:

Se cer:

Fig. 7.4

Indicație:

Se izolează punctul *D* al firului, se introduc tensiunile în cele două porțiuni de fir secționate şi se scriu ecuațiile de echilibru. Se separă porțiunea de fir *BD*, tăind firul în *D*, lângă punctul de fixare a

greutății Q. Se studiază echilibrul arcului de lănțișor BD, obserervând că punctul D nu se află pe axa Oy, deoarece înclinarea tangentei la fir în acest punct este nenulă.

Răspuns:

$$T_A = T_B = \frac{L}{8f}(2Q + pL) + \frac{1}{2}pf; T_D = \frac{L}{8f}(2Q + pL) - \frac{1}{2}pf;$$

$$\sin\alpha = \sin\beta = \frac{4f(pL + Q)}{2LQ + p\cdot\left(L^2 + 4f^2\right)}; \sin\theta = \frac{4f\cdot Q}{2LQ + p\cdot\left(L^2 - 4f^2\right)}.$$

CAPITOLUL 8

CINEMATICA PUNCTULUI MATERIAL

8.1. Generalități

8.1.1. Studiul mişcării punctului material în coordonate carteziene

Ecuațiile parametrice ale traiectoriei sunt:

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

Ecuația implicită se obține prin eliminarea timpului t din cele trei ecuații şi anume:

$$\begin{cases} f_1(x;y;z) = 0 \\ f_2(x;y;z) = 0 \end{cases}$$

Rezultă două suprafețe a căror intersecție este o curbă plană.
În plan ecuațiile parametrice ale traiectoriei punctului material arată astfel:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Eliminând timpul din ambele ecuații rezultă ecuația implicită a traiectoriei punctului material care este o curbă plană şi anume:

$$f(x;y)=0$$

Vectorial traiectoria unui punct material se scrie astfel:

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

Mărimea vectorului de mai sus se determină cu următoarea relație:

$$r = \sqrt{x^2 + y^2 + z^2}$$

Pentru viteze se scriu următoarele ecuații:

$$\begin{cases} v_x = \dot{x} = \dot{x}(t) \\ v_y = \dot{y} = \dot{y}(t) \Rightarrow f(\dot{x},\dot{y},\dot{z}) = 0 \\ v_z = \dot{z} = \dot{z}(t) \end{cases}$$

De asemenea expresia vectorială şi cea scalară a vitezei se scriu astfel:

$$\bar{v} = \dot{\bar{r}} = v_x\bar{i} + v_y\bar{j} + v_z\bar{k} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k}$$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

Pentru accelerații se scriu ecuațiile:

$$\begin{cases} a_x = \ddot{x} = \ddot{x}(t) \\ a_y = \ddot{y} = \ddot{y}(t) \Rightarrow f(\ddot{x},\ddot{y},\ddot{z}) = 0 \\ a_z = \ddot{z} = \ddot{z}(t) \end{cases}$$

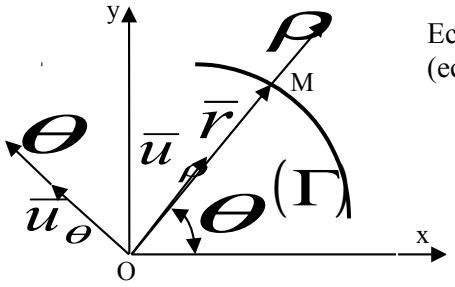
Expresia vectorială şi cea scalară pentru acceleraţii se scriu astfel:

$$\bar{a} = \ddot{\bar{r}} = a_x\bar{i} + a_y\bar{j} + a_z\bar{k} = \ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k}$$

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$$

8.1.2. Studiul mişcării punctului material în coordonate polare

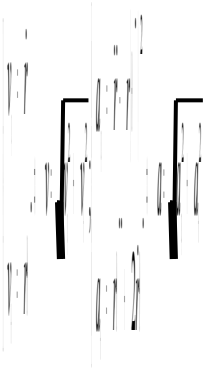
Nu sunt monitorizate decât traiectoriile plane



Ecuaţiile parametrice ale traiectoriei:
(ecuaţia traiectoriei în coordonate polare)

Fig. 8.1

Pentru viteze şi acceleraţii se scriu ecuaţiile:



8.1.3. Studiul mişcării punctului material în coordonate intrinseci

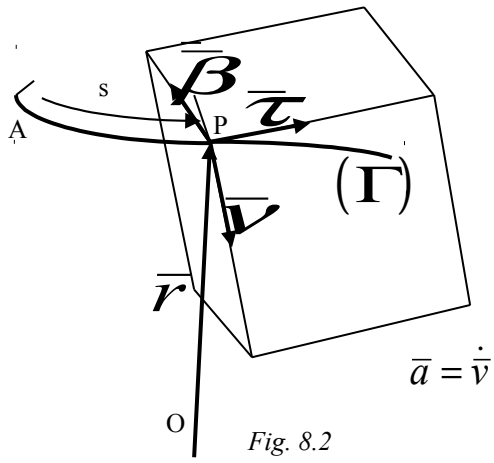


Fig. 8.2

$s = s(t)$ (ecuația orară a mișcării)

$$\bar{v} = \dot{\bar{s}} = v_\tau \bar{\tau} + v_\nu \bar{\nu} + v_\beta \bar{\beta} \Rightarrow \begin{cases} v_\tau = \dot{s} \\ v_\nu = 0 \\ v_\beta = 0 \end{cases}$$

$$\bar{a} = \dot{\bar{v}} = \ddot{\bar{s}} = \ddot{s}\bar{\tau} + \frac{\dot{s}^2}{\rho}\bar{\nu} = a_\tau \bar{\tau} + a_\nu \bar{\nu} + a_\beta \bar{\beta} \Rightarrow \begin{cases} a_\tau = \ddot{s} = \dot{v} \\ a_\nu = \dfrac{\dot{s}^2}{\rho} = \dfrac{v^2}{\rho} \\ a_\beta = 0 \end{cases}$$

8.2. Cinematica punctului material în coordonate carteziene

8.2.1. Cinematica punctului material liber

8.2.1.1. Probleme rezolvate

1)

Date: $\begin{cases} x = 2\cos 2t \\ y = \sin t \end{cases}$ ***Se cer:*** *Traiectoria mișcării punctului material;*
Viteza punctului material;
Accelerația punctului material.

Rezolvare:

$$\bar{r} = x\bar{i} + y\bar{j} = 2\cos 2t\bar{i} + \sin t\bar{j}; \cos 2t = 1 - 2\sin^2 t \Leftrightarrow \frac{x}{2} = 1 - 2y^2 \Leftrightarrow x = 2 - 4y^2 \, (\textit{parabolă})$$

$$\begin{cases} \dot{x} = -4\sin 2t \\ \dot{y} = \cos t \end{cases} \Rightarrow v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{16\sin^2 2t + \cos^2 t} \Rightarrow \bar{v} = \dot{x}\bar{i} + \dot{y}\bar{j} = -4\sin 2t\bar{i} + \cos t\bar{j}$$

$$\begin{cases} \ddot{x} = -8\cos 2t \\ \ddot{y} = -\sin t \end{cases} \Rightarrow a = \sqrt{\ddot{x}^2 + \ddot{y}^2} = \sqrt{64\cos^2 2t + \sin^2 t} \Rightarrow \bar{a} = \ddot{x}\bar{i} + \ddot{y}\bar{j} = -8\cos 2t\bar{i} - \sin t\bar{j}$$

2)

Date: $\begin{cases} x = \lambda + R\cos(\omega \cdot t) \\ y = R\sin(\omega \cdot t) \end{cases}$ ***Se cer:*** *Traiectoria mișcării punctului material;*
Viteza punctului material;
Accelerația punctului material.

$$\begin{cases} \dot{x} = -R\omega\sin\omega t \\ \dot{y} = R\omega\cos\omega t \end{cases} \Rightarrow v = \sqrt{\dot{x}^2 + \dot{y}^2} = R\omega \Rightarrow \bar{v} = \dot{x}\bar{i} + \dot{y}\bar{j} = -R\omega\sin\omega t\,\bar{i} - \cos\omega t\,\bar{j}$$

$$\begin{cases} \ddot{x} = -R\omega^2\cos\omega t \\ \ddot{y} = -R\omega^2\sin\omega t \end{cases} \Rightarrow a = \sqrt{\ddot{x}^2 + \ddot{y}^2} = R\omega^2 \Rightarrow \bar{a} = \ddot{x}\bar{i} + \ddot{y}\bar{j} = -R\omega^2\cos\omega t\,\bar{i} + \sin\omega t\,\bar{j}$$

$\dot{x} = -R\omega\sin\omega t$

$\ddot{x} = -R\omega^2\cos\omega t$

$\ddot{y} = -R\omega^2\sin\omega t$

$\bar{r} = x\bar{i} + y\bar{j} = [\lambda + R\cos(\omega \cdot t)] \cdot \bar{i} + R\sin(\omega \cdot t)\,\bar{j}; R^2 = (x-\lambda)^2 + y^2 \Rightarrow$ *cercuri*

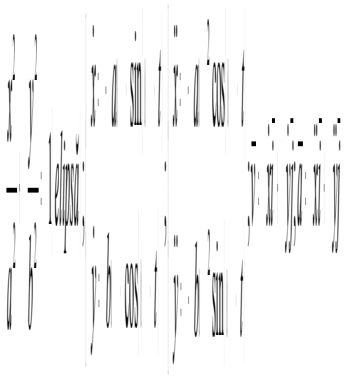
Rezolvare:

8.2.1.2. Probleme propuse

1)

Date: $\begin{cases} x = a \cdot \cos(\omega \cdot t) \\ y = b \cdot \sin(\omega \cdot t) \end{cases} a; b; \omega = ct$ ***Se cer:*** *Traiectoria mişcării punctului material;*
Viteza punctului material;
Acceleraţia punctului material.

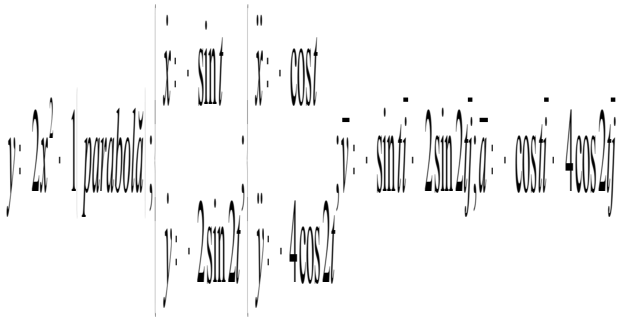
Răspuns:



2)

Date: $\begin{cases} x = \cos t \\ y = \cos 2t \end{cases}$ ***Se cer:*** *Traiectoria mişcării punctului material;*
Viteza punctului material;
Acceleraţia punctului material.

Răspuns:



8.2.2. Cinematica punctului material legat

8.2.2.1. Probleme rezolvate

1)

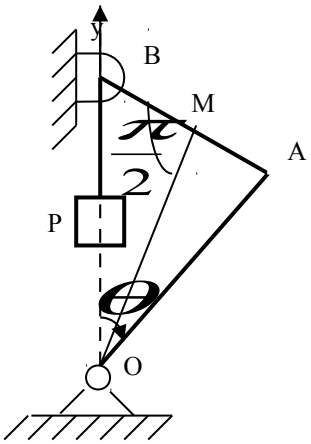


Fig. 8.3

Date:

Se cer:

Rezolvare:

$$y_P = OP = OB - PB = l - (L - AB) = l - L + 2l\sin\frac{\theta}{2};$$

$$MA = MB = \frac{AB}{2} = OA\sin\frac{\theta}{2} = l\sin\frac{\theta}{2} \Rightarrow AB = 2l\sin\frac{\theta}{2};$$

$$v_P = \dot{y}_P = 2l\frac{1}{2}\dot{\theta}\cdot\cos\frac{\theta}{2} = l\dot{\theta}\cdot\cos\frac{\theta}{2} = v_A\cos\frac{\theta}{2};$$

$$a_P = \dot{v}_P = \ddot{x}_P = l\ddot{\theta}\cos\frac{\theta}{2} - \frac{1}{2}l\dot{\theta}^2\sin\frac{\theta}{2}.$$

2)

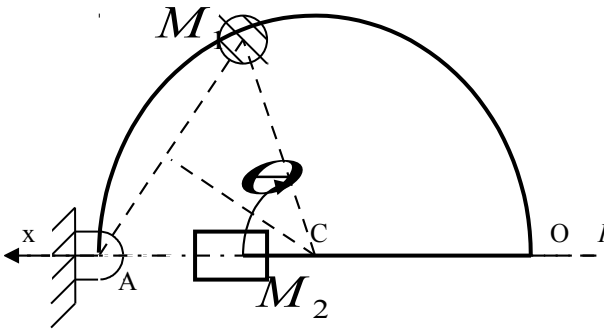


Fig. 8.4

Date:

Se cer:

Rezolvare:

$$x_{M2} = OM_2 = 2R - AM_2 = 2R - (2R - AM_1) = AM_1 = 2R\sin\frac{\theta}{2};$$

3)

$$\dot{\theta} = \frac{d\theta}{dt} = \omega_0 \Rightarrow d\theta = \omega_0 dt \Rightarrow \theta = \int \omega_0 dt = \omega_0 \int dt = \omega_0 t + C;$$

$$t = 0 \Rightarrow \theta = 0 \Rightarrow C = 0 \Rightarrow \theta = \omega_0 t;$$

$$v_{M2} = \dot{x}_{M2} = 2R\frac{\omega_0}{2}\cos\frac{\omega_0 t}{2} \Rightarrow a_{M2} = \dot{v}_{M2} = \ddot{x}_{M2} = -\frac{R}{2}\omega_0^2 \sin\frac{\omega_0 t}{2}.$$

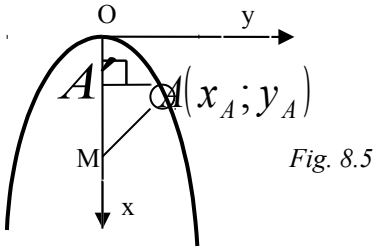


Fig. 8.5

Date:

Se cer:

Rezolvare:

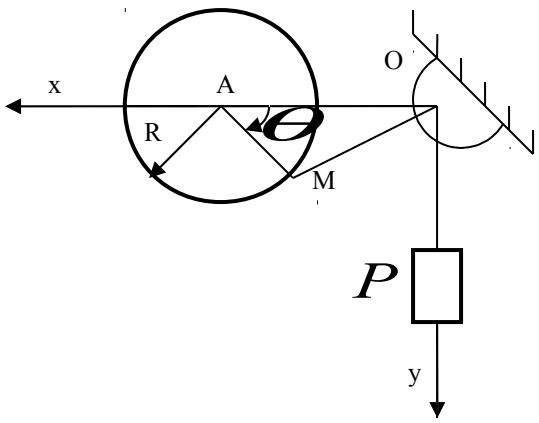
$$x_M = OM = OA + AM = x_A + \sqrt{l^2 - AA^2} = x_A + \sqrt{l^2 - y_A^2}; \text{ dar } : x_A = 2y_A^2 \Rightarrow \text{parametrizare}$$

$$\begin{cases} x_A = 2t^2 \\ y_A = t \end{cases} \Rightarrow x_M = 2t^2 + \sqrt{l^2 - t^2} \Rightarrow v_M = \dot{x}_M = 4t - \frac{t}{\sqrt{l^2 - t^2}};$$

$$a_M = \ddot{x}_M = \dot{v}_M = 4 - \left(\frac{\sqrt{l^2 - t^2} + t^2 \Big/ \sqrt{l^2 - t^2}}{l^2 - t^2} \right) = 4 - \frac{l^2}{\left(l^2 - t^2\right)\sqrt{l^2 - t^2}}.$$

8.2.2.2. Probleme propuse

1)



Date:

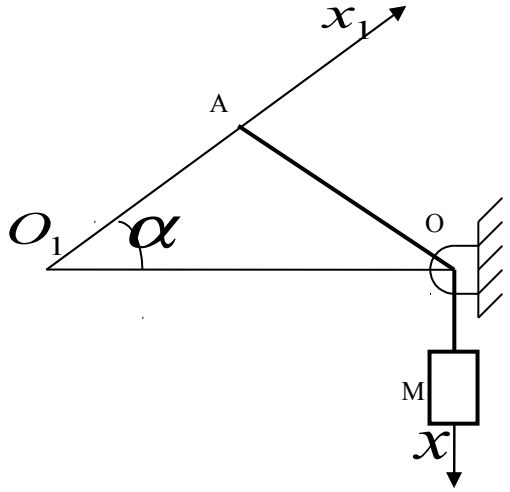
Se cer:

Fig. 8.6

Răspuns:

$$v_P = -R\omega_0 \frac{2\sin\theta}{\sqrt{5-4\cos\theta}}; a_P = -2R\omega_0^2 \frac{\cos\theta\cdot(5-4\cos\theta)+2\sin^2\theta}{(5-4\cos\theta)\sqrt{5-4\cos\theta}}.$$

2)



Date:

Se cer:

Fig. 8.7

Răspuns:

$$x_M = L - \sqrt{x_l^2 + l^2 - 2x_l l\cos\alpha}; v_M = -u\frac{x_l - l\cos\alpha}{\sqrt{x_l^2 + l^2 - 2x_l l\cos\alpha}}; a_M = -\frac{u^2 l^2 \sin^2\alpha}{\left(x_l^2 + l^2 - 2x_l l\cos\alpha\right)^{\frac{3}{2}}}.$$

8.2.3. Cinematica punctului material în alte sisteme de coordonate

8.2.3.1. Probleme rezolvate

1)

Date: $\begin{cases} r = 2 \\ \theta = \dfrac{1}{2}t^2 \end{cases}$

Se cer: *Traiectoria mişcării punctului material în coordonate polare;*
Viteza punctului material în coordonate polare;
Accelerația punctului material în coordonate polare.

Rezolvare:

Traiectoria punctului material este cercul de centru *O* şi de rază *r=2* deoarece *r=ct.*

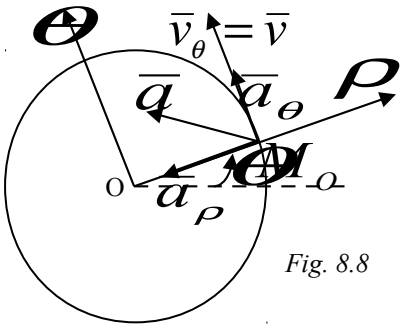


Fig. 8.8

$$\begin{cases} v_\rho = \dot{r} = 0 \\ v_\theta = r\dot{\theta} = 2t \end{cases} \Rightarrow v = \sqrt{v_\rho^2 + v_\theta^2} = 2t = v_\theta$$

$$\begin{cases} a_\rho = \ddot{r} - r\dot{\theta}^2 = -2t^2 \\ a_\theta = r\ddot{\theta} - 2\dot{r}\dot{\theta} = 2 \end{cases} \Rightarrow a = \sqrt{a_\rho^2 + a_\theta^2} = 2\sqrt{t^4 + 1}$$

2)

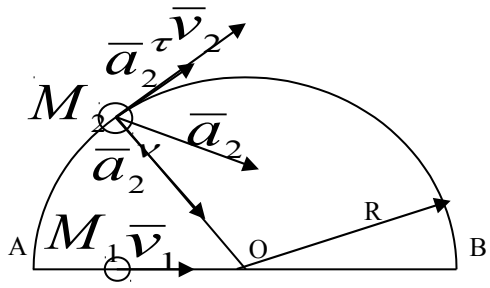


Fig. 8.9

Date:

Pornire simultană din A şi sosire simultană în B;

Se cer:

Rezolvare:

La pornirea din A:

Pentru primul corp:

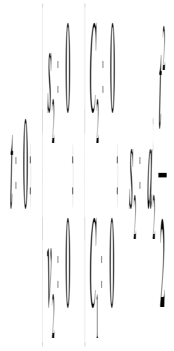
$$(M_1): v_1 = v_0 = \dot{s}_1 = ct \Rightarrow s_1 = \int v_1 dt = v_0 \int dt = v_0 t + C$$

$$t = 0 \Rightarrow s_1 = 0 \Rightarrow C = 0 \Rightarrow s_1 = v_0 t$$

Pentru al doilea corp:

$$(M_2): a_2^\tau = \dot{v}_2 = ct \Rightarrow v_2 = \int a_2^\tau dt = a_2^\tau t + C_1;$$

$$\dot{s}_2 = v_2 \Rightarrow s_2 = \int v_2 dt = \int (a_2^\tau t + C_1) dt = a_2^\tau \frac{t^2}{2} + C_1 t + C_2.$$



La întâlnirea în B:
Pentru primul corp:

$$(M_1): s_1 = 2R = v_0 t_B \Rightarrow t_B = \frac{2R}{v_0}$$

Pentru al doilea corp:

$$(M_2): s_2 = \pi \cdot R = a_2^\tau \frac{t_B^2}{2} \Rightarrow a_2^\tau = \frac{\pi \cdot v_0^2}{2R} \Rightarrow v_2 = a_2^\tau t = \frac{\pi \cdot v_0^2}{2R} t$$

$$v_2^B = a_2^\tau t_B = \frac{\pi \cdot v_0^2}{2R} \cdot \frac{2R}{v_0} = \pi \cdot v_0$$

$$a_2^\tau = \frac{v^2}{R} = \frac{\pi^2 v_0^4}{4R^3} t^2 \Rightarrow ct \Rightarrow a_{2B}^\tau = \frac{\pi^2 v_0^4}{4R^3} t_B^2 \Rightarrow$$

$$a_{2B}^\tau = \frac{\pi^2 v_0^4}{4R^3} t_B^2 = \frac{\pi^2 v_0^4}{4R^3} \cdot \frac{4R^2}{v_0^2} = \frac{\pi^2 v_0^2}{R} \Rightarrow$$

$$a_2 = \sqrt{\left(a_2^\tau\right)^2 + \left(a_2^n\right)^2}$$

$$a_{2B} = \sqrt{\left(a_{2B}^\tau\right)^2 + \left(a_{2B}^n\right)^2}$$

$$a_2 = \sqrt{\left(\frac{\pi^2 v_0^4}{4R^3} t^2\right)^2 + \left(\frac{\pi \cdot v_0^2}{2R}\right)^2}$$

$$a_{2B} = \sqrt{\frac{\pi^4 v_0^4}{R^2} + \frac{\pi^2 \cdot v_0^4}{4R^2}}$$

3)

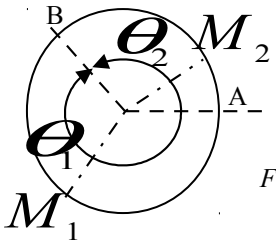


Fig. 8.10

Date:
Pornire simultană din A şi sosire simultană în B;

Se cer:
timpul şi locul întâlnirii.

Rezolvare:

La pornire:
Corpul 1:

$$(M_1): v_1 = v_0 = ct \Rightarrow s_1 = \int v_1 dt = v_0 \int dt = v_0 t + C$$
$$t = 0 \Rightarrow s_1 = 0 \Rightarrow C = 0 \Rightarrow s_1 = v_0 t$$

Corpul 2:

$$(M_2): a_2^\tau < 0 = ct \Rightarrow -a_2^\tau = ct = \dot{v}_2 = -a_2^\tau \Big| dt = -a_2^\tau t + C_1 \Rightarrow$$

$$\Rightarrow s_2 = \int v_2 dt = -a_2^\tau \frac{t^2}{2} + C_1 t + C_2$$

$$t = 0 \Rightarrow \begin{cases} s_2 = 0 \\ v_2 = v_0 \end{cases} \Rightarrow \begin{cases} C_2 = 0 \\ C_1 = v_0 \end{cases} \Rightarrow s_2 = a_2^\tau \frac{t^2}{2} + v_0 t$$

La sosirea în B:

Corpul 1:

$$(M_1): s_1 = R\theta_1 = v_0 t_B$$

Corpul 2:

$$(M_2): s_2 = R\theta_2 = -a_2^\tau \frac{t_B^2}{2} + v_0 t_B;$$
$$\theta_1 + \theta_2 = 2\pi \Leftrightarrow \theta_1 = 2\pi - \theta_2;$$
$$v_2 = 0 \Rightarrow 0 = -a_2^\tau t_B + v_0;$$

Rezultă:

$$\theta_2 = \frac{1}{R}\left(v_0 t_B - a_2^\tau \frac{t_B^2}{2}\right) \Rightarrow \theta_1 = 2\pi - \frac{1}{R}\left(v_0 t_B - a_2^\tau \frac{t_B^2}{2}\right);$$

$$t_B = \frac{R\theta_1}{v_0} = \frac{R}{v_0}\left[2\pi - \frac{1}{R}\left(v_0 t_B - a_2^\tau \frac{t_B^2}{2}\right)\right] = \frac{2\pi R}{v_0} - \frac{1}{v_0}\left(v_0 t_B - a_2^\tau \frac{t_B^2}{2}\right) = \frac{2\pi R}{v_0} - t_B + a_2^\tau \frac{t_B^2}{2v_0}$$

$$a_2^\tau \frac{t_2^2}{2v_0} - 2t_B + \frac{2\pi R}{v_0} = 0 \Rightarrow \frac{v_0}{t_B}\cdot\frac{t_B^2}{2v_0} - 2t_B + \frac{2\pi R}{v_0} \Rightarrow \frac{t_B}{2} - 2t_B + \frac{2\pi R}{v_0} = 0 \Rightarrow \frac{3}{2}t_B = \frac{2\pi R}{v_0} \Rightarrow$$

$$\Rightarrow t_B = \frac{4\pi R}{3v_0}; a_2^\tau = \frac{v_0}{t_B} = \frac{3v_0^2}{4\pi R}; \theta_1 = \frac{4\pi}{3}; \theta_2 = \frac{2\pi}{3}; v_2 = v_0 - a_2^\tau t = v_0 - \frac{3v_0^2}{4\pi R}t \Rightarrow a_2^\nu = \frac{v_2^2}{R} \Rightarrow$$

$$\Rightarrow a_2 = \sqrt{\left(a_2^\nu\right)^2 + \left(a_2^\tau\right)^2} = \sqrt{\frac{v_2^4}{R^2} + \frac{9v_0^4}{16\pi^2 R^2}}.$$

8.2.3.2. Probleme propuse

1)

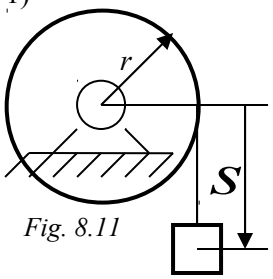


Fig. 8.11

Date:

Se cer:
pentru t=1s

Răspuns:

2)

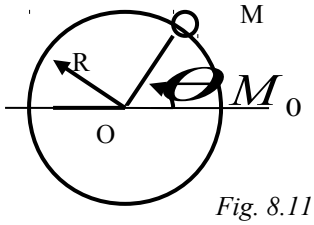


Fig. 8.11

Date:

Se cer:

Indicații:

Răspuns:

$$\omega = 4t; \varepsilon = 4; v = 4Rt; a_\nu = 16Rt^2; a_\tau = 4R \Rightarrow a = R\sqrt{256t^4 + 16} = 4R\sqrt{16t^4 + 1}.$$

3)

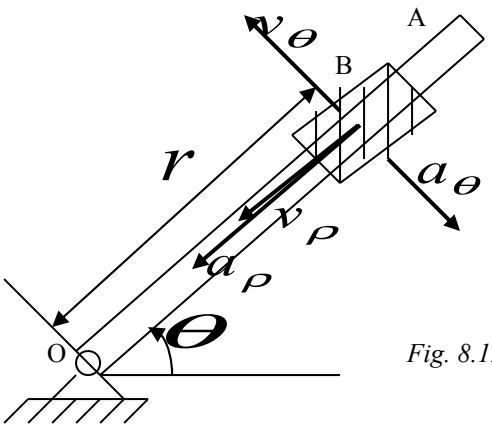


Fig. 8.12

Date:

Se cer:

Răspuns:

CAPITOLUL 9

CINEMATICA SOLIDULUI RIGID

9.1. Generalități. Mişcările simple ale rigidului

9.1.1. Translația

Este mişcarea în timpul căreia orice dreaptă a rigidului rămâne paralelă cu ea însăşi.

$$\begin{cases} \bar{v} = \bar{v}_0 \\ \bar{a} = \bar{a}_0 \end{cases}$$

9.1.2. Rotația

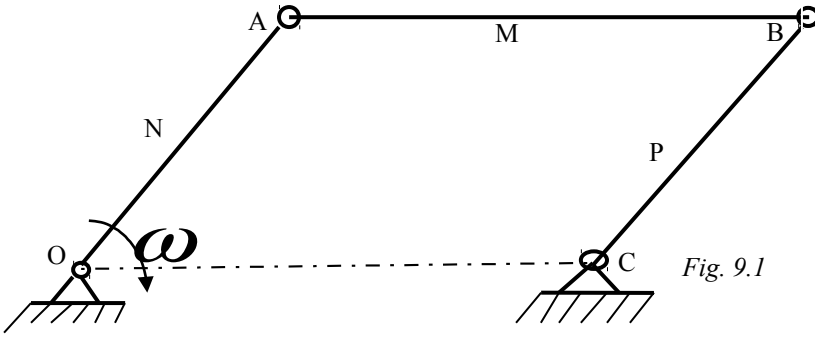
Este mişcarea în care două puncte distincte ale rigidului (adică o dreaptă a sa), rămân fixe tot timpul mişcării.

$$\begin{cases} \bar{v} = \bar{\omega} \times \bar{r} \\ \bar{a} = \bar{\varepsilon} \times \bar{r} + \bar{\omega} \times \left(\bar{\omega} \times \bar{r} \right) \end{cases}$$

Fie o axă în jurul căreia se roteşte corpul. Atunci fiecare punct are mişcare circulară. Vectorii viteză unghiulară şi respectiv acceleraţie unghiulară au drept suport axa de rotaţie.

9.2. Probleme rezolvate

1)



Date:

Se cer:
Vitezele şi acceleraţiile punctelor:
A; B; M; N; P.

Fig. 9.1

Rezolvare:

$$a_A = a_B = a_M = a = \sqrt{\left(a^\tau\right)^2 + \left(a^\nu\right)^2} = l\sqrt{\varepsilon^2 + \omega^4} = l\sqrt{4 + 16t^4} = 2l\sqrt{1 + 4t^4};$$

$$a_N = a_P = \frac{a}{2} = \sqrt{\left(a_N^\tau\right)^2 + \left(a_N^\nu\right)^2} = \frac{l}{2}\sqrt{\varepsilon^2 + \omega^4} = \frac{l}{2}\sqrt{4 + 16t^4} = l\sqrt{1 + 4t^4}.$$

Determinarea vitezelor:

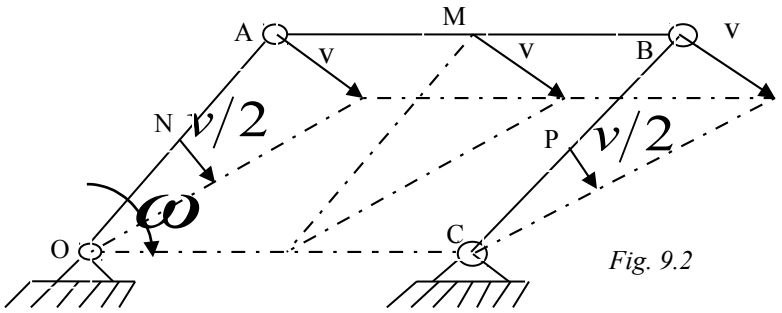


Fig. 9.2

$$v_A = v_M = v_B = l\omega = 2tl = v;$$

$$v_N = v_P = \frac{l}{2}\omega = tl. = \frac{v}{2}.$$

Determinarea acceleraţiilor:

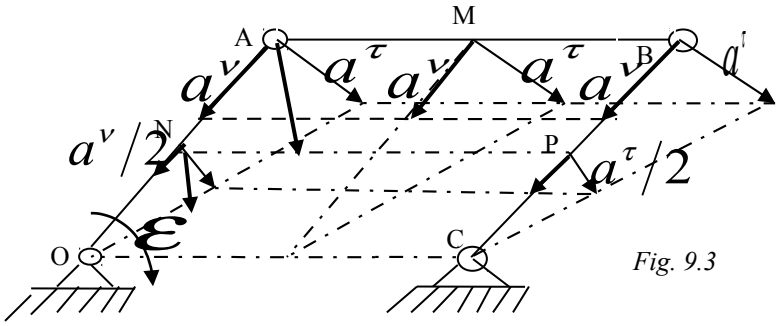


Fig. 9.3

$$a_A^\tau = a_B^\tau = a_M^\tau = a^\tau = l\varepsilon = 2l;$$

$$a_N^\tau = a_P^\tau = \frac{l}{2}\varepsilon = l = \frac{a^\tau}{2};$$

$$a_A^\nu = a_B^\nu = a_M^\nu = a^\nu = l\omega^2 = 4lt^2;$$

$$a_N^\nu = a_P^\nu = \frac{a^\nu}{2} = \frac{l}{2}\omega^2 = 2lt^2;$$

2)

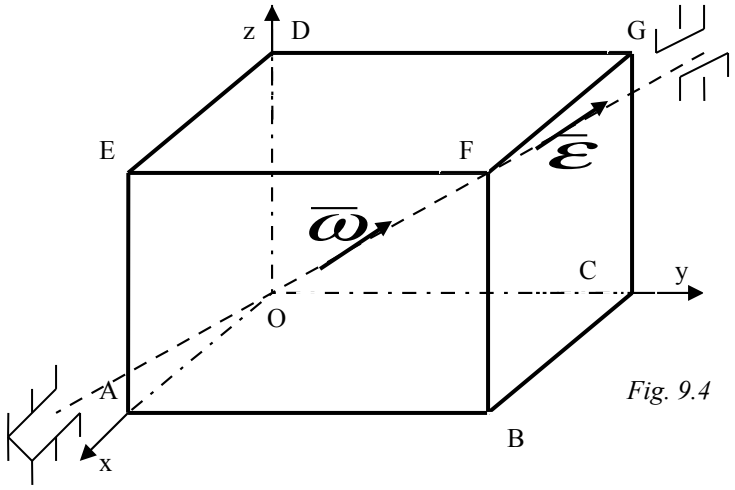


Fig. 9.4

Date:

$OA = 0{,}1m; OC = 0{,}2m;$

$OD = 0{,}3m; \theta = \frac{\sqrt{14}}{2}t^2.$

Se cer:

$\bar{v}_A; \bar{v}_B; \bar{v}_C; ...; \bar{v}_G = ?;$

$\bar{a}_A; \bar{a}_B; \bar{a}_C; ...; \bar{a}_G = ?$

Pentru t=2s.

Rezolvare:

$$\bar{\omega}=\omega\cdot vers(\overline{OF})=\theta\cdot\frac{\overline{OF}}{OF}=t\sqrt{14}\cdot\frac{0,1\bar{i}+0,2\bar{j}+0,3\bar{k}}{\sqrt{14}\cdot 10^{-1}}=t\bar{i}+2t\bar{j}+3t\bar{k};\ \omega t=2s=2\bar{i}+4\bar{j}+6\bar{k}.$$

$$\bar{v}_C=\bar{\omega}\times\overline{OC}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ x_C&y_C&z_C\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ 0&0,2&0\end{vmatrix}=-1,2\bar{i}+0,4\bar{k};$$

$$\bar{v}_A=\bar{\omega}\times\overline{OA}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ x_A&y_A&z_A\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ 0,1&0&0\end{vmatrix}=0,6\bar{j}-0,4\bar{k};$$

$$\bar{a}_C=\bar{\varepsilon}\times\overline{OC}+\bar{\omega}\times(\bar{\omega}\times\overline{OC})=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \varepsilon_x&\varepsilon_y&\varepsilon_z\\ x_C&y_C&z_C\end{vmatrix}+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ v_{Cx}&v_{Cy}&v_{Cz}\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 1&2&3\\ 0&0,2&0\end{vmatrix}+$$

$$\bar{\varepsilon}=\dot{\bar{\omega}}=\bar{i}+2\bar{j}+3\bar{k},\ sau:\ \bar{\varepsilon}=\varepsilon\frac{\overline{OF}}{OF}=\bar{i}+2\bar{j}+3\bar{k};$$

$$+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ -1,2&0&-0,4\end{vmatrix}=\bar{i}+6,4\bar{j}+4,8\bar{k};$$

$$\bar{a}_A=\bar{\varepsilon}\times\overline{OA}+\bar{\omega}\times(\bar{\omega}\times\overline{OA})=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \varepsilon_x&\varepsilon_y&\varepsilon_z\\ x_A&y_A&z_A\end{vmatrix}+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ v_{Ax}&v_{Ay}&v_{Az}\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 1&2&3\\ 0,1&0&0\end{vmatrix}+$$

$$\bar{v}_D=\bar{\omega}\times\overline{OD}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ x_D&y_D&z_D\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ 0&0&0,3\end{vmatrix}=1,2\bar{i}-0,6\bar{j};$$

$$+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ 0&0,6&-0,4\end{vmatrix}=-5,2\bar{i}+1,1\bar{j}+\bar{k};$$

$$\bar{a}_D=\bar{\varepsilon}\times\overline{OD}+\bar{\omega}\times(\bar{\omega}\times\overline{OD})=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \varepsilon_x&\varepsilon_y&\varepsilon_z\\ x_D&y_D&z_D\end{vmatrix}+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ v_{Dx}&v_{Dy}&v_{Dz}\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 1&2&3\\ 0&0&0,3\end{vmatrix}+$$

$$\bar{v}_B=\bar{\omega}\times\overline{OB}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ x_B&y_B&z_B\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ 0,1&0,2&0\end{vmatrix}=-1,2\bar{i}+0,6\bar{j};$$

$$+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ 1,2&-0,6&0\end{vmatrix}=4,2\bar{i}+6,9\bar{j}-6\bar{k};$$

$$\bar{a}_B=\bar{\varepsilon}\times\overline{OB}+\bar{\omega}\times(\bar{\omega}\times\overline{OB})=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \varepsilon_x&\varepsilon_y&\varepsilon_z\\ x_B&y_B&z_B\end{vmatrix}+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ v_{Bx}&v_{By}&v_{Bz}\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 1&2&3\\ 0,1&0,2&0\end{vmatrix}+$$

$$\bar{v}_E=\bar{\omega}\times\overline{OE}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ x_E&y_E&z_E\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ 0,1&0&0,3\end{vmatrix}=1,2\bar{i}-0,4\bar{k};$$

$$+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ -1,2&0,6&0\end{vmatrix}=-4,2\bar{i}-6,9\bar{j}+6\bar{k};$$

$$\bar{a}_E=\bar{\varepsilon}\times\overline{OE}+\bar{\omega}\times(\bar{\omega}\times\overline{OE})=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \varepsilon_x&\varepsilon_y&\varepsilon_z\\ x_E&y_E&z_E\end{vmatrix}+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ v_{Ex}&v_{Ey}&v_{Ez}\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 1&2&3\\ 0,1&0&0,3\end{vmatrix}+$$

$$+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ 1,2&0&-0,4\end{vmatrix}=-\bar{i}+8\bar{j}-5\bar{k};$$

Deoarece punctul *F* se găseşte pe axa de rotaţie iar vectorii viteză unghiulară şi acceleraţie unghiulară sunt coliniari cu axa de rotaţie atunci viteza şi acceleraţia unghiulară sunt nule adică:

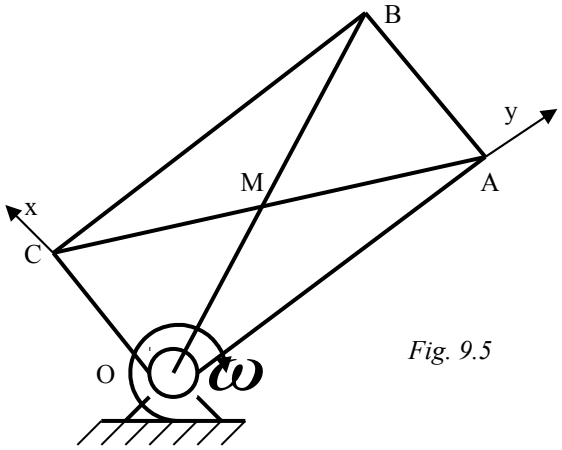
$$\bar{v}_F=0;\ \bar{a}_F=0.$$

$$\bar{v}_G=\bar{\omega}\times\overline{OG}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ x_G&y_G&z_G\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ 0&0,2&0,3\end{vmatrix}=-0,6\bar{j}+0,4\bar{k};$$

$$\bar{a}_G=\bar{\varepsilon}\times\overline{OG}+\bar{\omega}\times(\bar{\omega}\times\overline{OG})=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \varepsilon_x&\varepsilon_y&\varepsilon_z\\ x_G&y_G&z_G\end{vmatrix}+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ \omega_x&\omega_y&\omega_z\\ v_{Gx}&v_{Gy}&v_{Gz}\end{vmatrix}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 1&2&3\\ 0&0,2&0,3\end{vmatrix}+$$

$$+\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\ 2&4&6\\ 0&-0,6&0,4\end{vmatrix}=-5,2\bar{i}-1,1\bar{j}+1,4\bar{k}.$$

3)



Date:

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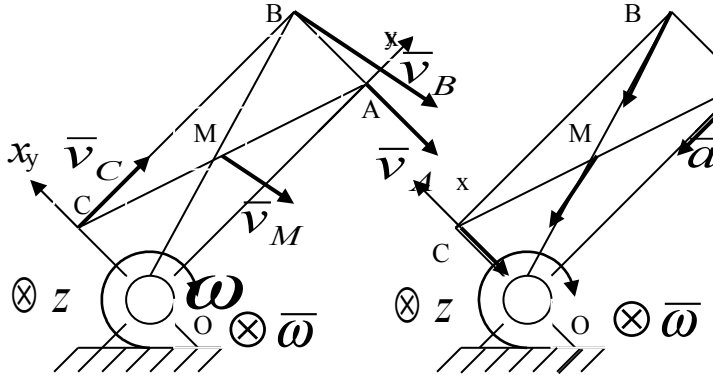
Fig. 9.5

Rezolvare:



$\omega = ct \Rightarrow \dot{\omega} = \varepsilon = 0 \Rightarrow$

$\Rightarrow a_A^\tau = a_B^\tau = a_C^\tau = a_M^\tau = 0;$

$\bar{a}_A^v = \bar{a}_A; \bar{a}_B^v = \bar{a}_B;$

$\bar{a}_C^v = \bar{a}_C; \bar{a}_M^v = \bar{a}_M.$

Fig. 9.6

$$\bar{v}_A = \bar{\omega} \times \overline{OA} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ x_A & y_A & z_A \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \omega_0 \\ 0 & 4 & 0 \end{vmatrix} = -4\omega_0 \cdot \bar{i};$$

$$\bar{v}_B = \bar{v}_A + \bar{\omega} \times \overline{AB} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ x_B & y_B & v_{Az} \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \omega_0 \\ 3 & 4 & 0 \end{vmatrix} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \omega_0 \\ -4\omega_0 & 0 & 0 \end{vmatrix} = -4\omega_0 \bar{i} + 3\omega_0 \bar{j} - 4\omega_0^2 \bar{j};$$

$$\bar{a}_B = \bar{\omega} \times (\bar{\omega} \times \overline{OB}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ v_{Bx} & v_{By} & v_{Bz} \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \omega_0 \\ -4\omega_0 & 3\omega_0 & 0 \end{vmatrix} = -3\omega_0^2 \bar{i} - 4\omega_0^2 \bar{j};$$

$$\bar{v}_C = \bar{\omega} \times \overline{OC} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ x_C & y_C & z_C \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \omega_0 \\ 3 & 0 & 0 \end{vmatrix} = 3\omega_0 \bar{j};$$

$$\bar{v}_M = \bar{\omega} \times \overline{OM} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ x_M & y_M & z_M \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \omega_0 \\ \frac{3}{2} & 2 & 0 \end{vmatrix} = -2\omega_0 \bar{i} + \frac{3}{2}\omega_0 \bar{j};$$

$$\bar{a}_C = \bar{\omega}\times(\bar{\omega}\times\overline{OC}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ v_{Cx} & v_{Cy} & v_{Cz} \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \omega_0 \\ 0 & 3\omega_0 & 0 \end{vmatrix} = -3\omega_0^2\bar{i};$$

4)

$$\bar{a}_M = \bar{\omega}\times(\bar{\omega}\times\overline{OM}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ v_{Mx} & v_{My} & v_{Mz} \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \omega_0 \\ -2\omega_0 & \frac{3}{2}\omega_0 & 0 \end{vmatrix} = -\frac{3}{2}\omega_0^2\bar{i} - 2\omega_0^2\bar{j}.$$

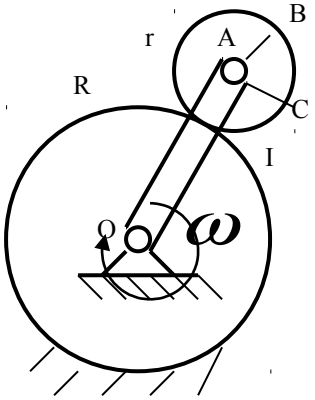


Fig. 9.7

Date:
rostogolire fără alunecare;

Se cer:
pentru discul mobil;

Rezolvare:

Centrul instantaneu de rotație este în punctul I de tangență între cele două discuri.

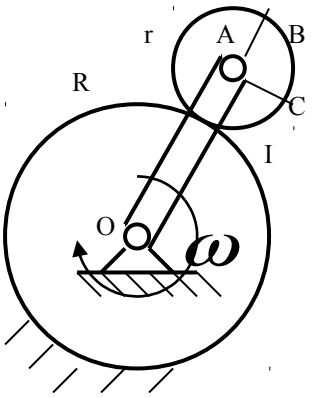


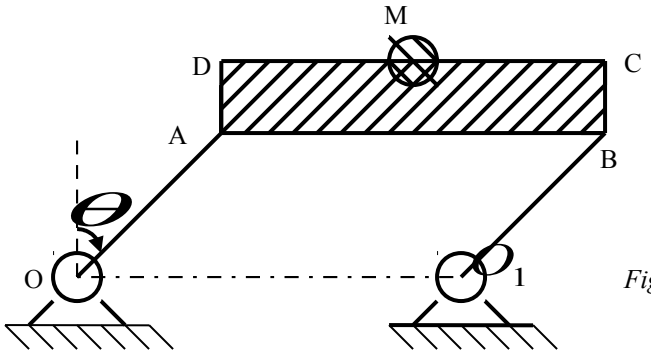
Fig. 9.8

$$v_A = \omega\cdot(R+r) = \Omega\cdot r \Rightarrow \Omega = \frac{R+r}{r}\omega \Rightarrow \mathrm{E} = 0 \Rightarrow$$

$$\Rightarrow v_B = 2v_A = 2\omega\cdot(R+r); v_C = IC\cdot\Omega = r\sqrt{2}\cdot\frac{R+r}{r}\omega = (R+r)\omega\sqrt{2};$$

$$a_A = (R+r)\omega^2.$$

9.3. Probleme propuse
1)



Date:

Se cer:

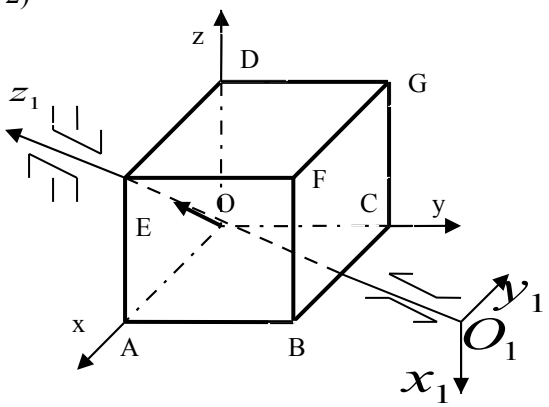
Fig. 9.9

Răspuns:

$$v_A = v_B = v_M = l\omega;$$

$$\varepsilon = 0 \Rightarrow a_A^\tau = a_B^\tau = a_M^\tau = 0; a_A = a_A^v = a_B = a_B^v = a_M = a_M^v = l\omega^2.$$

2)



Date:

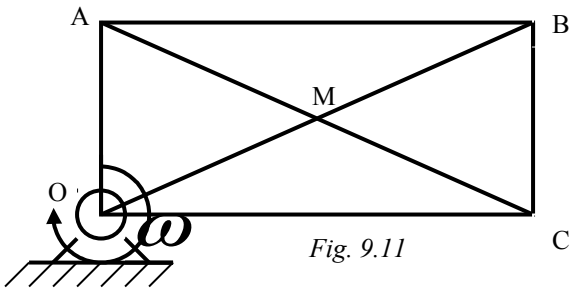
Se cer:

Fig. 9.10

Răspuns:

$\bar{v}_A = -2,4\bar{j}; \bar{v}_B = -2,4\bar{i} - 1,4\bar{j} - 1,8\bar{k}; \bar{v}_C = -2,4\bar{i} - 1,8\bar{k}; \bar{v}_D = -2,4\bar{j}; \bar{v}_F = -2,4\bar{i} - 1,8\bar{k};$
$\bar{v}_G = -2,4\bar{i} - 2,4\bar{j} - 1,8\bar{k}; \bar{a}_A = -19,2\bar{i} - 14,4\bar{k}; \bar{a}_B = -19,2\bar{i} - 30\bar{j} - 14,4\bar{k}; \bar{a}_C = -30\bar{j};$
$\bar{a}_D = -19,2\bar{i} - 14,\bar{k}; \bar{a}_F = -30\bar{j}; \bar{a}_G = -19,2\bar{i} - 30\bar{j} - 14,4\bar{k}.$

3)



Date:

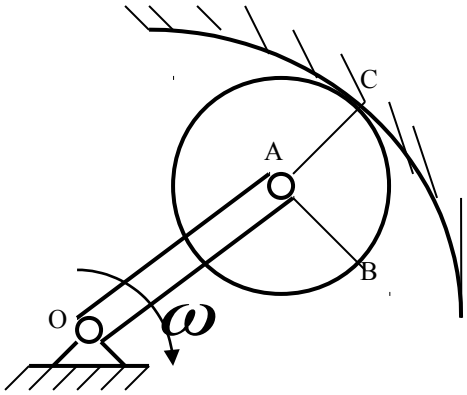
Se cer:

Fig. 9.11

Răspuns:

$$\bar{v}_A = 1,2\pi\bar{j}; \bar{a}_A = 4,8\pi^2\bar{i}; \bar{v}_B = -1,6\pi\bar{i} + 1,2\pi\bar{j}; \bar{a}_B = -4,8\pi^2\bar{i} - 6,4\pi^2\bar{j}; \bar{v}_C = -1,6\pi\bar{i}; \bar{a}_C = -6,4\pi^2\bar{j};$$
$$\bar{v}_M = -0,8\pi\bar{i} + 0,6\pi\bar{j}; \bar{a}_M = -2,4\pi^2\bar{i} - 3,2\pi^2\bar{j}.$$

4)



Date:
rostogolire fără alunecare.

Se cer:
pentru discul mobil;

Fig. 9.12

Răspuns:

$$CIR \equiv I \equiv C; v_A = (R-r)\cdot\omega = r\cdot\Omega \Rightarrow \Omega = \frac{R-r}{r}\omega \Rightarrow \mathrm{E} = 0; v_B = \Omega CB = \Omega r\sqrt{2} = (R-r)\omega\sqrt{2};$$

$$a_B = CB\Omega^2 = r\sqrt{2}\,\frac{(R-r)^2}{r^2}\omega^2.$$

CAPITOLUL 10

MIŞCAREA PLAN – PARALELĂ A SOLIDULUI RIGID

10.1. Definiții. Metoda CIR

Un rigid realizează o mişcare plan – paralelă dacă trei puncte necoliniare ale acestuia şi deci planul determinat de acestea rămân tot timpul mişcării într+un plan fix din spațiu.

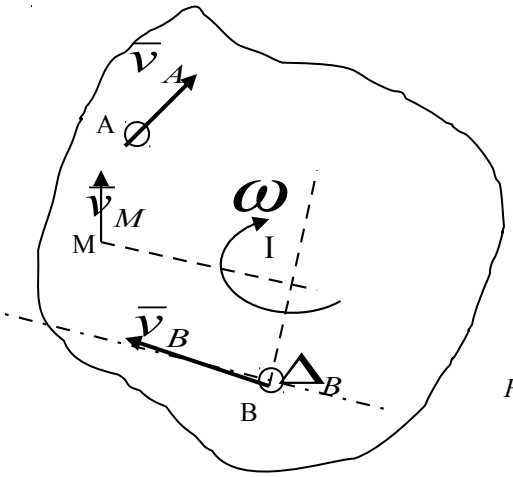
Centrul instantaneu de rotație (CIR=I) este locul geometric al punctelor din planul mobil cu viteză instantanee nulă. O proprietate importantă a CIR este aceea că în raport cu el vitezele tuturor punctelor din planul mobil se determină ca în mişcarea de rotație.

Pentru a se determina acest punct este nevoie de două elemente şi anume:

a) viteza unui punct al rigidului (modul, direcție şi sens);
b) suportul vitezei altui punct al rigidului.

Determinarea CIR se face astfel:

Se ridică perpendiculare pe suporturile vitezelor celor două puncte în punctele respective (fig. 10.1). La intersecția acestor perpendiculare se găseşte centrul instantaneu de rotație (CIR).



$$\omega = \frac{v_A}{IA};$$

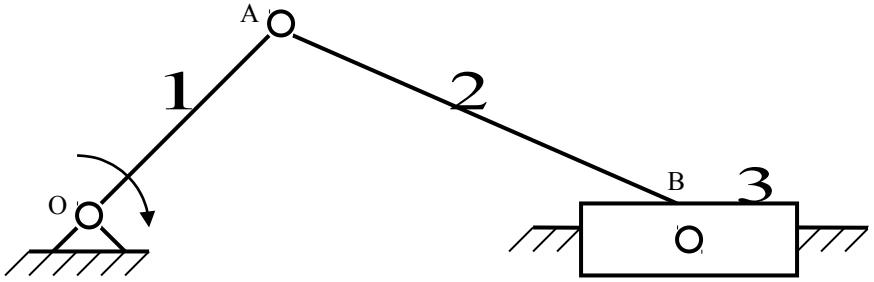
$$v_B = IB\omega = \frac{IB}{IA}\omega;$$

$$v_M = IM\omega = \frac{IM}{IA}v_A.$$

Fig. 10.1

10.1.1. Probleme rezolvate

1)



Date:
toate elementele geometrice.

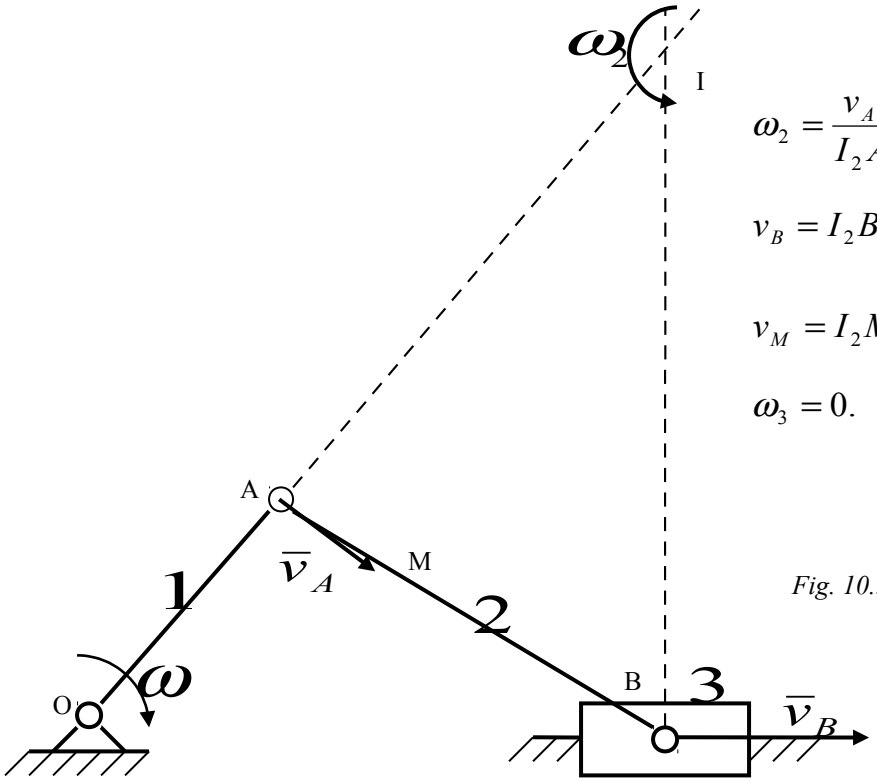
Se cer:

Fig. 10.2

Rezolvare:

Manivela 1 are mişcare de rotaţie (R), biela 2 are mişcare plan – paralelă (PP) iar pistonul 3 are mişcare de translaţie (T).

$$v_A = OA\omega$$



$$\omega_2 = \frac{v_A}{I_2 A} = \frac{OA}{I_2 A}\omega;$$

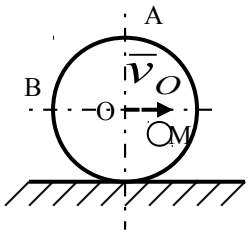
$$v_B = I_2 B \cdot \omega_2 = \frac{I_2 B}{I_2 A} OA\omega;$$

$$v_M = I_2 M \cdot \omega_2 = \frac{I_2 M}{I_2 A} OA\omega;$$

$$\omega_3 = 0.$$

Fig. 10.3

2)



Date:

Se cer:

Fig. 10.4

Observație:
Mereu în cazul roților care au rostogolire fără alunecare, (RFA), centrul instantaneu de rotație (CIR) se găsește în punctul de contact al roții cu calea de rulare.

Rezolvare:

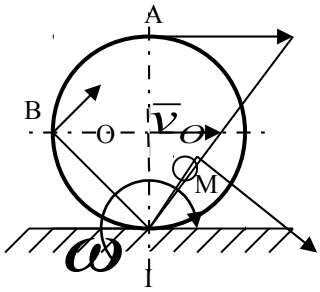
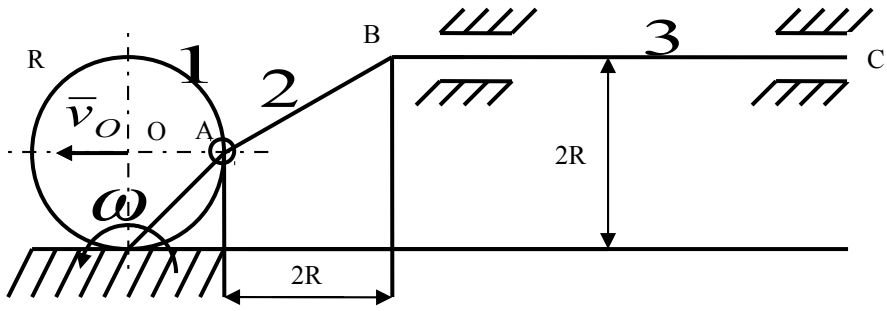


Fig. 10.5

3)



Date:

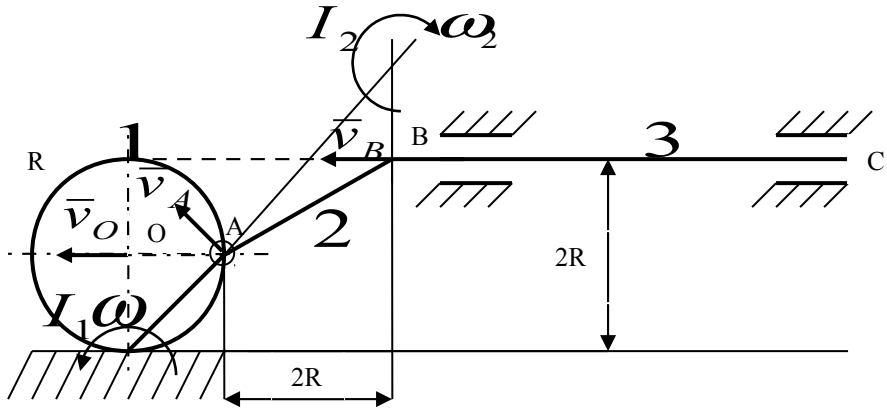
Se cer:

Fig. 10.6

Rezolvare:

$$\omega = \frac{v_O}{I_1 O} = \frac{v_O}{R}; v_A = I_1 A\omega = \frac{I_1 A}{R} v_O = \frac{R\sqrt{2}}{R} v_O = v_O\sqrt{2}; \omega_2 = \frac{v_A}{I_2 A} = \frac{v_O\sqrt{2}}{2R\sqrt{2}} = \frac{v_O}{2R}; \omega_3 = 0;$$

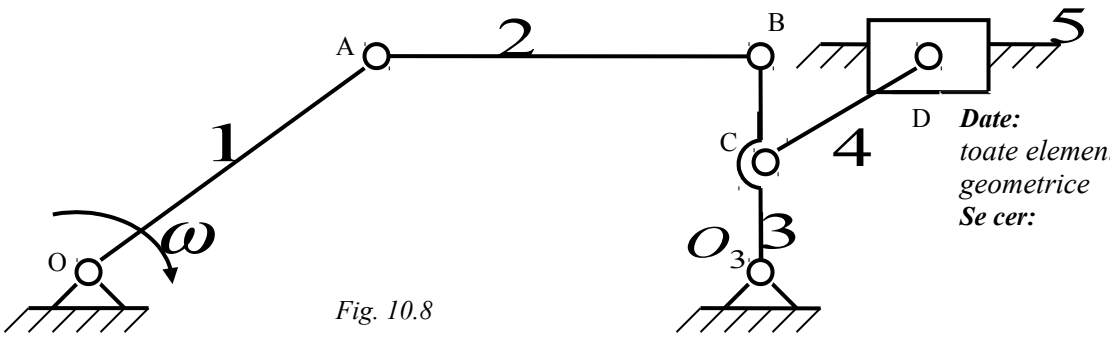
$$v_B = v_C = I_2 B\omega_2 = \frac{v_O}{2R} R = \frac{v_O}{2}.$$



1 – plan paralelă;
2 – plan paralelă;
3 – translație.

Fig. 10.7

4)



Date:
toate elementele geometrice
Se cer:

Fig. 10.8

Rezolvare:

Din figura 10.8 rezultă că: elementul *1* are mişcare de rotaţie, elementul *2* are mişcare plan – paralelă, elementul *3* are mişcare de rotaţie, elementul *4* are mişcare plan – paralelă, elementul *5* are mişcare de translaţie.

$$v_A = \omega \cdot OA$$

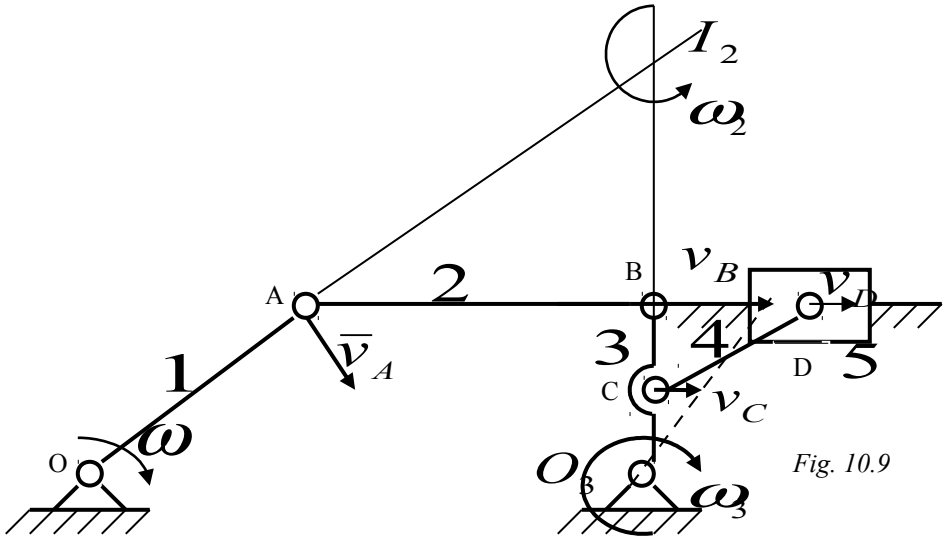


Fig. 10.9

$$\omega_2 = \frac{v_A}{I_2 A} = \frac{OA}{I_2 A}\omega; v_B = I_2 B \cdot \omega_2 = \frac{I_2 B}{I_2 A} OA\omega; \omega_3 = \frac{v_B}{BO_3} = \frac{I_2 B}{I_2 A} \cdot \frac{OA}{BO_3}\omega;$$

$$v_C = O_3 C\omega_3 = \frac{I_2 B}{I_2 A} \cdot \frac{O_3 C}{O_3 B} \cdot OA\omega; I_4 \to \infty \Rightarrow \omega_4 = 0 \Rightarrow 4 = translatie$$

5)

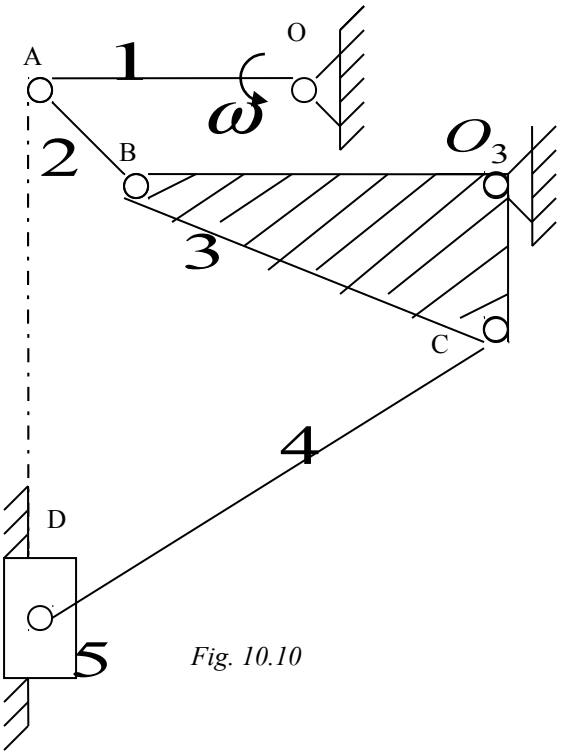
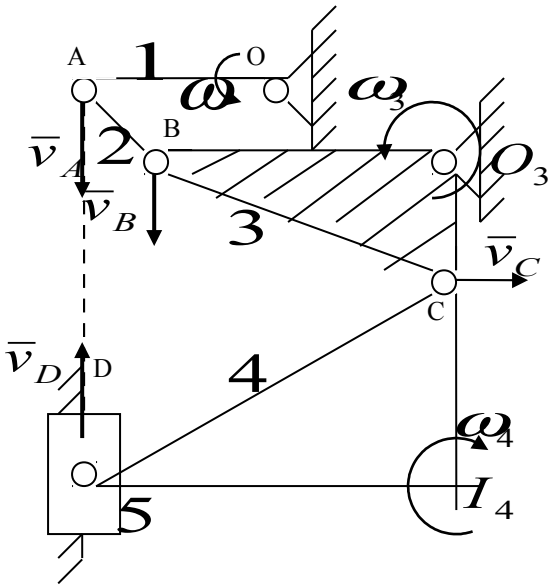


Fig. 10.10

Date:
toate elementele geometrice.
Se cer:



Rezolvare:

Elementul *1* are mişcare de rotație, elementul *2* are mişcare plan – paralelă, elementul *3* are mişcare de rotație, elementul *4* are mişcare plan – paralelă, elementul *5* are mişcare de translație.

$$v_A = OA\omega,$$

$$I_2 \to \infty \Rightarrow \omega_2 = 0 \Rightarrow 2 = translatie \Rightarrow v_B = v_A = OA\omega; \omega_3 = \frac{v_B}{BO_3} = \frac{AO}{BO_3}\omega;$$

$$v_C = CO_3\omega_3 = \frac{CO_3}{BO_3}OA\omega; \omega_4 = \frac{v_C}{CI_4} = \frac{CO_3}{BO_3}\cdot\frac{OA}{CI_4}\omega; v_D = I_4D\omega_4 = \frac{I_4D}{I_4C}\cdot\frac{CO_3}{BO_3}OA\omega; \omega_5 = 0.$$

10.1.2. Probleme propuse

1)

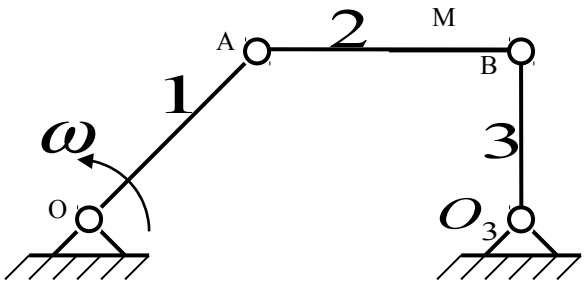


Fig. 10.11

Date:
toate elementele geometrice.

Se cer:

Indicație:

$$v_A = OA\omega; \omega_2 = \frac{v_A}{I_2 A}; v_B = I_2 B\omega_2; v_M = I_2 M \cdot \omega_2; \omega_3 = \frac{v_B}{O_3 B}.$$

2)

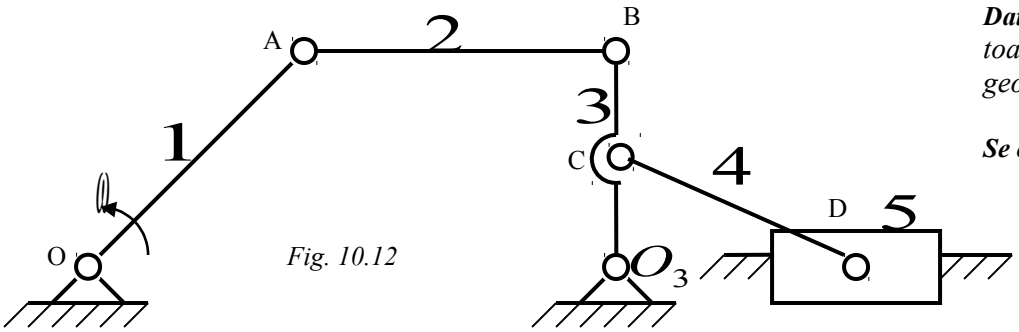


Fig. 10.12

Date:
toate elementele geometrice

Se cer:

Indicație:

$$v_A = OA\omega; \omega_2 = \frac{v_A}{I_2 A}; v_B = I_2 B \cdot \omega_2; \omega_3 = \frac{v_B}{BO_3}; v_C = CO_3 \cdot \omega_3; \omega_4 = 0; \omega_5 = 0.$$

3)

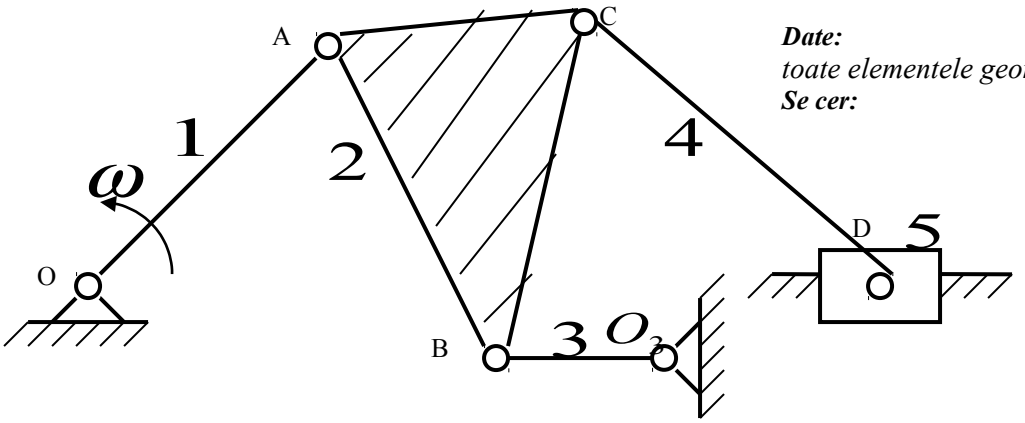


Fig. 10.13

Date:
toate elementele geometrice.
Se cer:

Indicație:

$$v_A = OA\omega; \omega_2 = \frac{v_A}{I_2 A}; v_B = I_2 B \cdot \omega_2; v_C = I_2 C \cdot \omega_2; \omega_3 = \frac{v_B}{BO_3}; \omega_4 = \frac{v_C}{CI_4}; v_D = I_4 D \cdot \omega_4; \omega_5 = 0.$$

10.2. Metoda planului vitezelor şi a ecuațiilor vectoriale. Determinarea accelerațiilor

O altă metodă de determinare a distribuției vitezelor pentru un mecanism plan este metoda planului vitezelor şi a ecuațiilor vectoriale. Este o metodă grafo – analitică prin care se rezolvă grafic ecuații vectoriale de tip Euler.

$$\begin{cases} \bar{v}_B = \bar{v}_A + \bar{\omega} \times \overline{AB} \\ \bar{\omega} \times \overline{AB} \perp \overline{AB} \end{cases} \Rightarrow \begin{cases} \bar{v}_B = \bar{v}_A + \bar{v}_{BA} \\ \bar{v}_{BA} \perp \overline{AB} \end{cases}$$

Pe desen, distribuția vitezelor se reprezintă la scară cu ajutorul coeficientului de scară pentru viteze:

$$k_v \left[\frac{m/s}{mm} \right]$$

Pentru determinarea distribuției accelerațiilor se foloseşte metoda planului accelerațiilor şi a ecuațiilor vectoriale care este, de asemenea, o metodă grafo – analitică prin care se rezolvă grafic ecuații vectoriale de tip Rivals.

$$\begin{cases} \bar{a}_B = \bar{a}_A + \bar{\omega} \times \overline{AB} + \bar{\omega} \times (\bar{\omega} \times \overline{AB}) \\ \bar{\omega} \times (\bar{\omega} \times \overline{AB}) \parallel \overline{BA} \\ \bar{\varepsilon} \times \overline{AB} \perp \overline{AB} \end{cases} \Rightarrow \begin{cases} \bar{a}_B = \bar{a}_A + \bar{a}_{BA}^n + \bar{a}_{BA}^t \\ \bar{a}_{BA}^n \parallel \overline{BA} \quad \bar{a}_{BA}^t \perp \overline{BA} \\ \bar{a}_{BA}^t \perp \overline{BA} \end{cases}$$

Pe desen, distribuția accelerațiilor se reprezintă la scară cu ajutorul coeficientului de scară pentru accelerații:

$$k_a \left[\frac{m/s^2}{mm} \right]$$

Pentru simplificarea calculelor se consideră: $k_v=1$ şi $k_a=1$.

10.2.1. Probleme rezolvate

1)

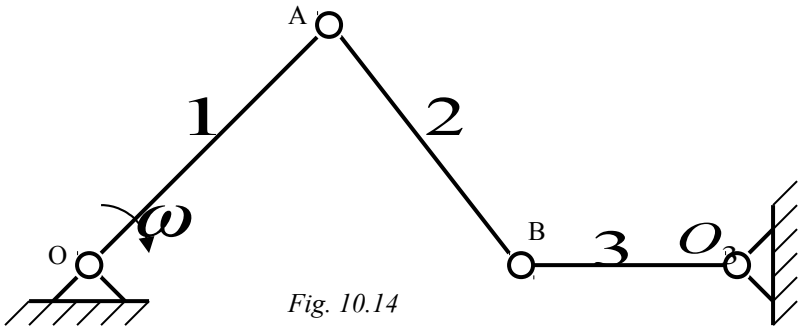


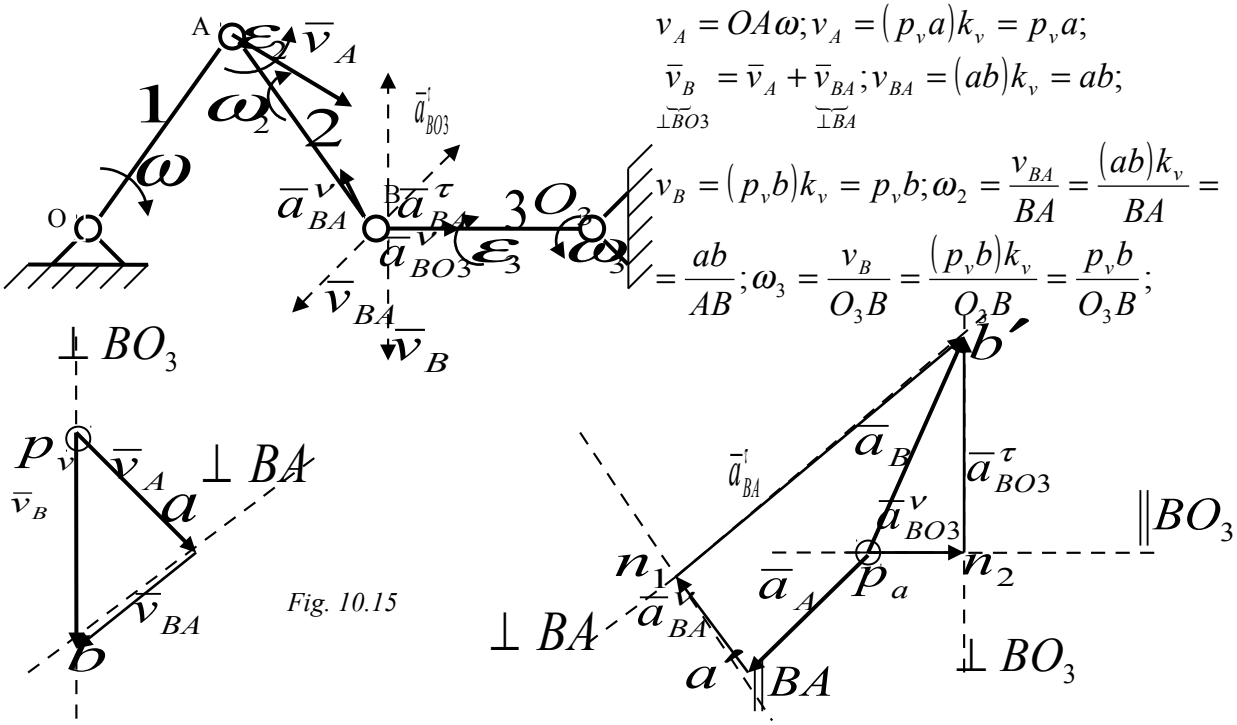
Fig. 10.14

Date:
toate elementele geometrice

Se cer:

Rezolvare:

Elementul *1* are mişcare de rotație, elementul *2* are mişcare plan – paralelă, elementul *3* are mişcare de rotație.



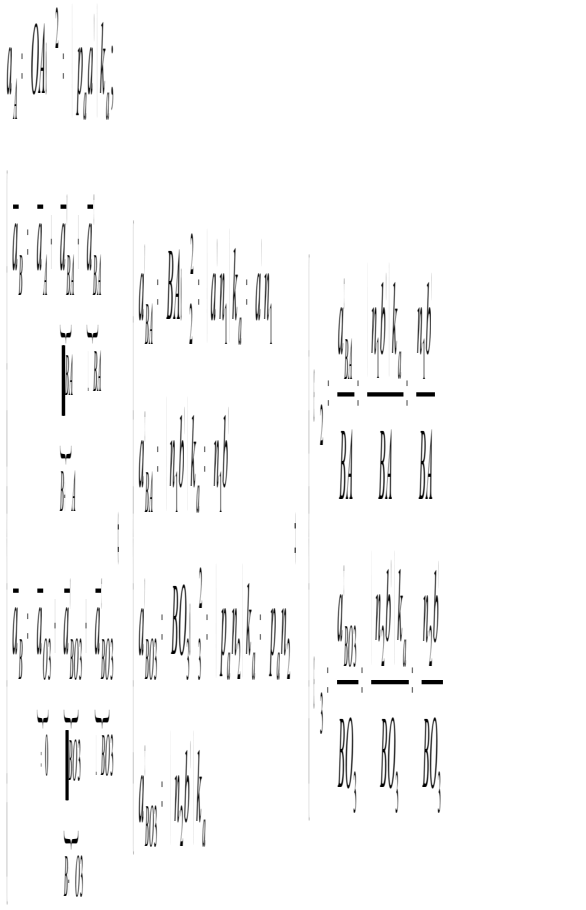
$$v_A = OA\omega; v_A = (p_v a)k_v = p_v a;$$

$$\underset{\perp BO3}{\bar{v}_B} = \underset{}{\bar{v}_A} + \underset{\perp BA}{\bar{v}_{BA}}; v_{BA} = (ab)k_v = ab;$$

$$v_B = (p_v b)k_v = p_v b; \omega_2 = \frac{v_{BA}}{BA} = \frac{(ab)k_v}{BA} =$$

$$= \frac{ab}{AB}; \omega_3 = \frac{v_B}{O_3 B} = \frac{(p_v b)k_v}{O_3 B} = \frac{p_v b}{O_3 B};$$

Fig. 10.15



2)

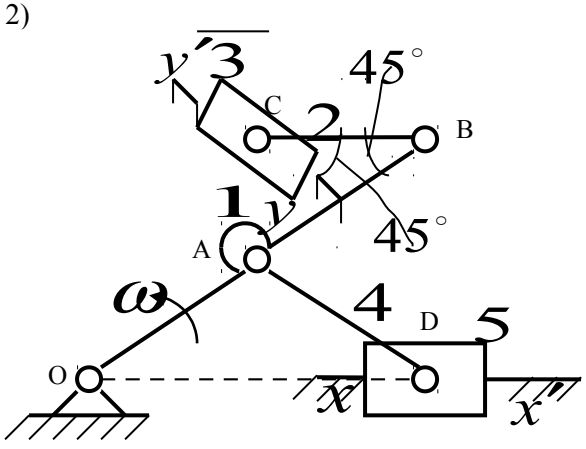


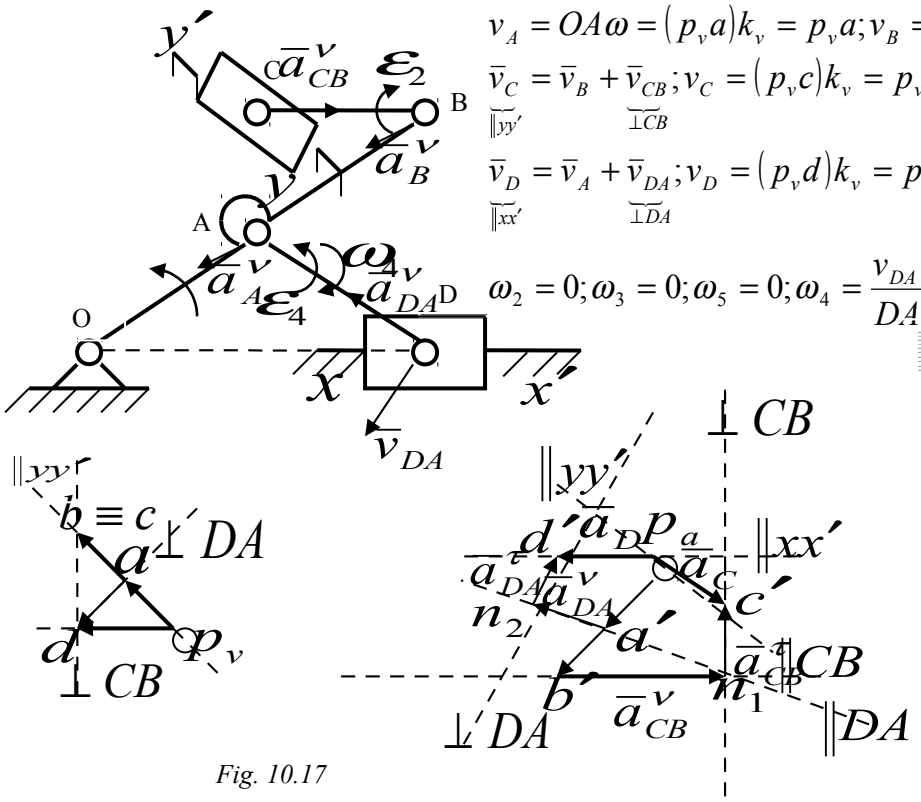
Fig. 10.16

Date:
toate elementele geometrice

Se cer:

Rezolvare:

Elementul *1* are mişcare de rotaţie, elementul *2* are mişcare plan – paralelă, elementul *3* are mişcare de translaţie, elementul *4* are mişcare plan – paralelă, elementul *5* are mişcare de translaţie.



$$v_A = OA\omega = (p_v a)k_v = p_v a; v_B = OB\omega = (p_v b)k_v = p_v b;$$

$$\underset{\|yy'}{\bar{v}_C} = \underset{}{\bar{v}_B} + \underset{\perp CB}{\bar{v}_{CB}}; v_C = (p_v c)k_v = p_v c; v_{CB} = (cb)k_v = cb;$$

$$\underset{\|xx'}{\bar{v}_D} = \underset{}{\bar{v}_A} + \underset{\perp DA}{\bar{v}_{DA}}; v_D = (p_v d)k_v = p_v d; v_{DA} = (da)k_v = da;$$

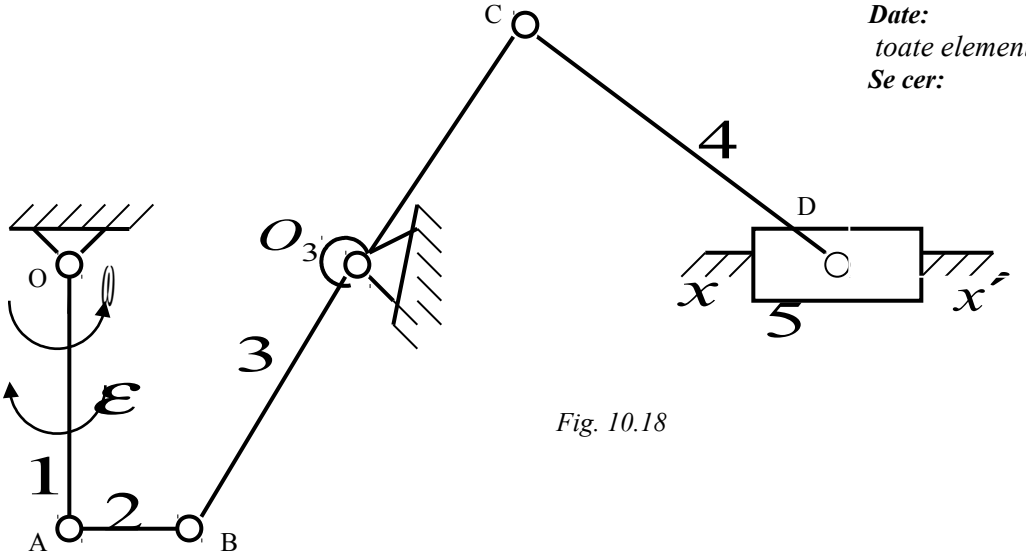
$$\omega_2 = 0; \omega_3 = 0; \omega_5 = 0; \omega_4 = \frac{v_{DA}}{DA} = \frac{(da)k_v}{DA} = \frac{da}{DA}.$$

Fig. 10.17

$$\underset{\|yy'}{\bar{a}_C} = \underset{}{\bar{a}_B} + \underset{\|CB}{\bar{a}^{\nu}_{CB}} + \underset{\perp CB}{\bar{a}^{\tau}_{CB}} \Rightarrow \varepsilon_2 = \frac{a^{\tau}_{CB}}{CB} = \frac{n_1 c'}{CB}; \underset{\|xx'}{\bar{a}_D} = \underset{}{\bar{a}_A} + \underset{\|DA}{\bar{a}^{\nu}_{DA}} + \underset{\perp DA}{\bar{a}^{\tau}_{DA}} \Rightarrow \varepsilon_4 = \frac{a^{\tau}_{DA}}{DA} = \frac{n_2 d'}{DA}; \varepsilon_3 = 0; \varepsilon_5 = 0.$$

$\overset{}{C \to B}$ $\overset{}{D \to A}$

3)



Date:
toate elementele geometrice
Se cer:

Fig. 10.18

Rezolvare:

Elementul *1* are mişcare de rotaţie, elementul *2* are mişcare plan – paralelă, elementul *3* are mişcare de rotaţie, elementul *4* are mişcare plan – paralelă, elementul *5* are mişcare de translaţie.

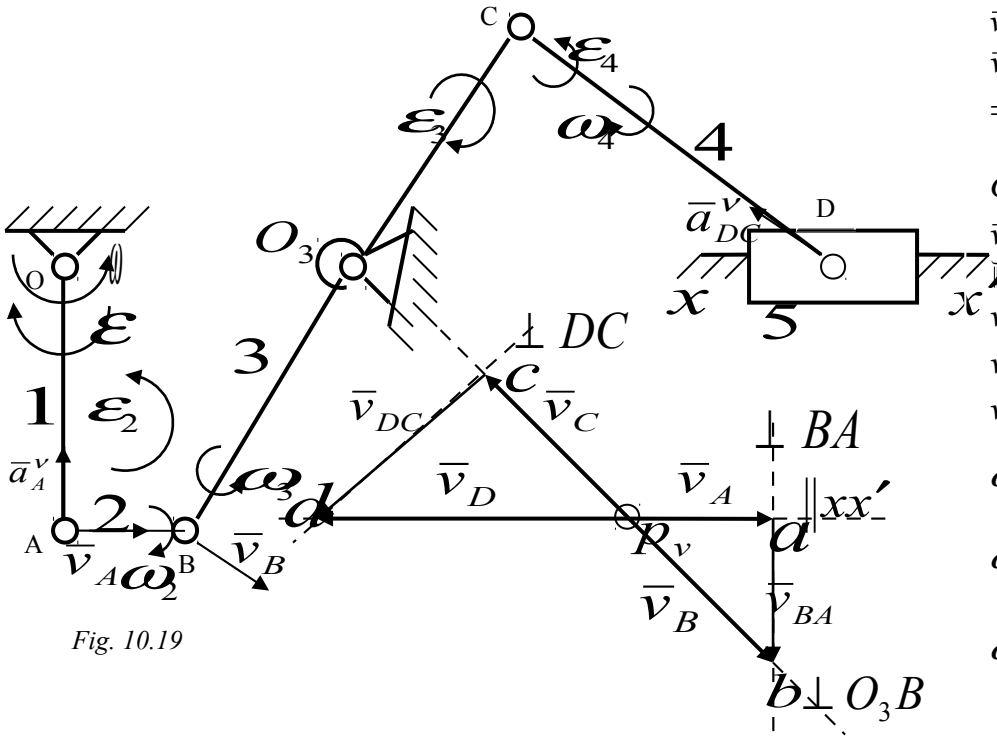
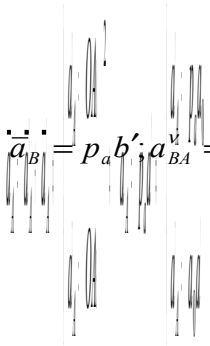
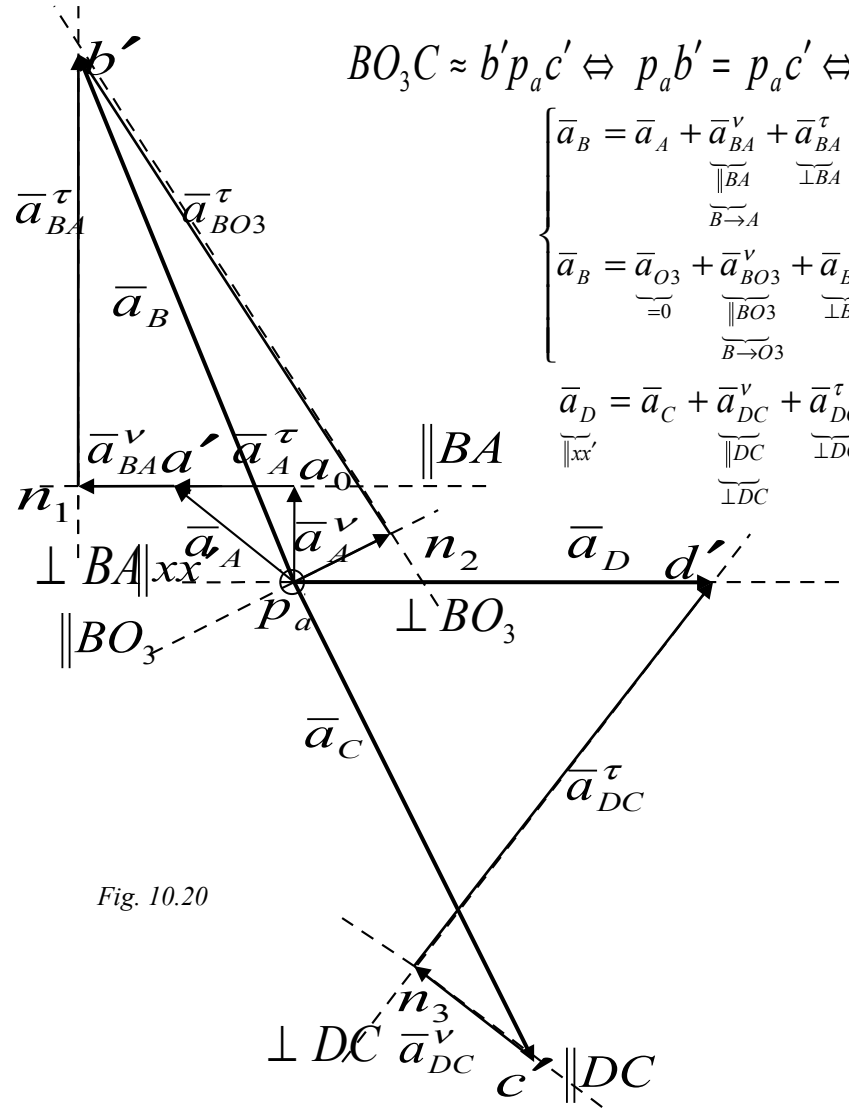


Fig. 10.19

$$\bar{v}_A = OA\omega$$
$$\bar{v}_B = \bar{v}_A + \bar{v}_{BA} \Rightarrow$$
$$\Rightarrow \underset{\perp BO_3}{\bar{v}_B} = \bar{v}_A + \underset{\perp BA}{\bar{v}_{BA}};$$
$$CO_3B \cong \bar{c}p_v\bar{b}; \Rightarrow \bar{v}_C;$$
$$\bar{v}_D = \bar{v}_C + \underset{\perp DC}{\bar{v}_{DC}};$$
$$\underset{xx'}{\bar{v}_D}$$
$$v_A = p_v a; v_{BA} = ab;$$
$$v_B = p_v b; v_C = p_v c;$$
$$v_D = p_v d; v_{DC} = cd;$$
$$\omega_2 = \frac{v_{BA}}{BA} = \frac{ab}{BA};$$
$$\omega_3 = \frac{v_B}{O_3B} = \frac{p_v b}{O_3B};$$
$$\omega_4 = \frac{v_{DC}}{DC} = \frac{cd}{DC};$$



$$\bar{a}_B = p_a b'; a^{\nu}_{BA} = BA\omega_2^2 = a'n_1; a^{\tau}_{BA} = n_1 b'; a^{\nu}_{BO3} = p_a n_2; a^{\tau}_{BO3} = n_2 b'; a_C = p_a c'$$



$$BO_3C \approx b'p_ac' \Leftrightarrow p_ab' = p_ac' \Leftrightarrow a_B = a_C;$$

$$\begin{cases} \bar{a}_B = \bar{a}_A + \underbrace{\underbrace{\bar{a}^{\nu}_{BA}}_{\parallel BA}}_{B\to A} + \underbrace{\bar{a}^{\tau}_{BA}}_{\perp BA} \\ \bar{a}_B = \underbrace{\bar{a}_{O3}}_{=0} + \underbrace{\underbrace{\bar{a}^{\nu}_{BO3}}_{\parallel BO3}}_{B\to O3} + \underbrace{\bar{a}_{BO3}}_{\perp BO3} \end{cases}$$

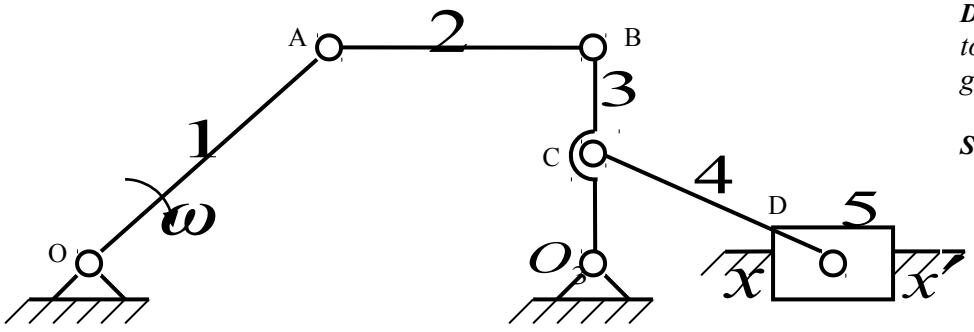
$$\bar{a}_D = \underbrace{\bar{a}_C}_{\parallel xx'} + \underbrace{\bar{a}^{\nu}_{DC}}_{\parallel DC} + \underbrace{\bar{a}^{\tau}_{DC}}_{\perp DC}$$

$$a_D = p_ad';$$
$$a^{\nu}_{DC} = c'n_3;$$
$$a^{\tau}_{DC} = n_3d';$$

$$\varepsilon_2 = \frac{a^{\tau}_{BA}}{BA} = \frac{n_1b'}{BA};$$
$$\varepsilon_3 = \frac{a^{\tau}_{BO3}}{BO_3} = \frac{n_2b'}{BO_3};$$
$$\varepsilon_4 = \frac{a^{\tau}_{DC}}{DC} = \frac{n_3d'}{DC};$$
$$\varepsilon_5 = 0.$$

Fig. 10.20

4)



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Se cer:

Fig. 10.21

Rezolvare:

Elementul *1* are mişcare de rotaţie, elementul *2* are mişcare plan – paralelă, elementul *3* are mişcare de rotaţie, elementul *4* are mişcare plan – paralelă, elementul *5* are mişcare de translaţie.

$$v_A = OA\omega = (p_va)k_v = p_va; a_A = a^{\nu}_A = OA\omega^2 = (p_aa')k_a$$

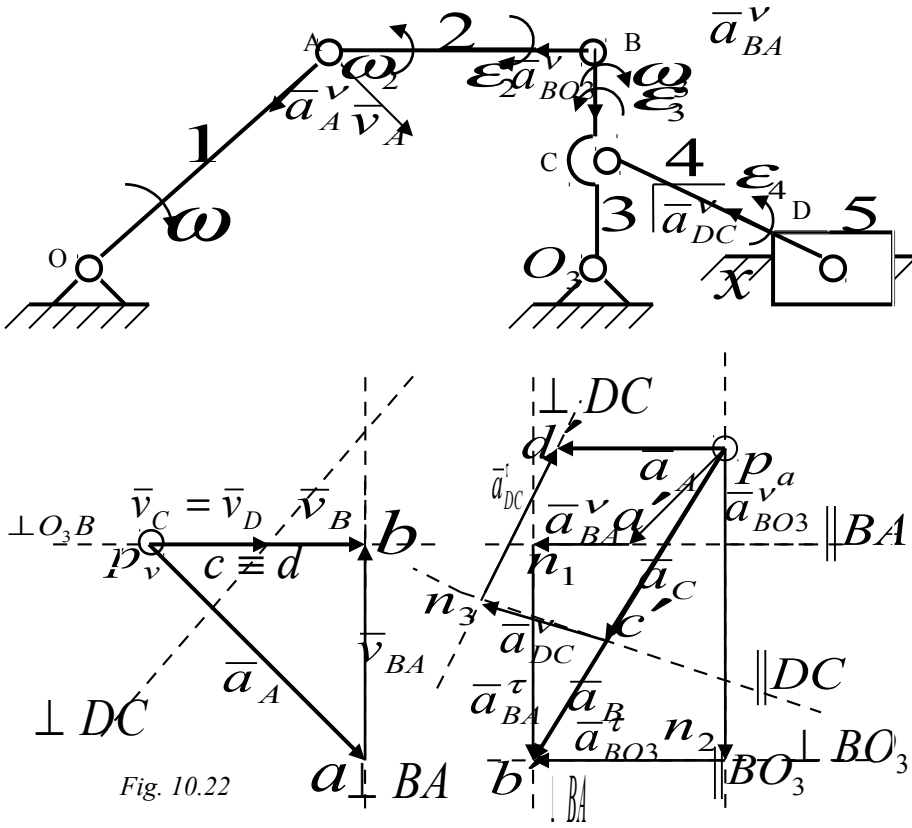

Fig. 10.22

$$\underset{\perp \overline{BO3}}{\bar{v}_B} = \underset{\perp \overline{BA}}{\bar{v}_A + \bar{v}_{BA}};$$

$$v_B = (p_v b)k_v = p_v b;$$

$$v_{BA} = (ab)k_v = ab;$$

$$BCO_3 \approx bcp_v \Rightarrow \frac{v_C}{v_B} = \frac{CO_3}{BO_3} \Rightarrow v_C = \frac{CO_3}{BO_3} v_B;$$

$$\underset{\parallel \overline{xx'}}{\bar{v}_D} = \bar{v}_C + \underset{\perp \overline{DC}}{\bar{v}_{DC}};$$

$$v_C = (p_v c)k_v = p_v c = v_D = (p_v d)k_v = p_v d;$$

$$\omega_2 = \frac{v_{BA}}{BA} = \frac{(ab)k_v}{BA} = \frac{ab}{BA};$$

$$\omega_3 = \frac{v_B}{BO_3} = \frac{(p_v b)k_v}{BO_3} = \frac{p_v b}{BO_3}; \omega_5 = 0; \omega_4 = 0.$$

$$\begin{cases} \bar{a}_B = \bar{a}_A + \bar{a}^{n}_{BA} + \bar{a}^{\tau}_{BA} \\ \quad\quad \| BA \quad \perp BA \\ \quad\quad B \to A \\ \bar{a}_B = \bar{a}_{O3} + \bar{a}^{n}_{BO3} + \bar{a}^{\tau}_{BO3} \\ \quad\quad =0 \quad \| BO3 \quad \perp BO3 \\ \quad\quad B \to O3 \end{cases} \Rightarrow \begin{cases} a_B = |p_a b'| k_a = p_a b'; a^{n}_{BA} = BA\omega_2^2 = |a' n_1| k_a = a' n_1; \\ a^{\tau}_{BA} = |n_1 b'| k_a = n_1 b'; \varepsilon_2 = \dfrac{a^{\tau}_{BA}}{BA} = \dfrac{|n_1 b'| k_a}{BA} = \dfrac{n_1 b'}{BA}; \\ a^{n}_{BO3} = BO_3\omega_3^2 = |p_a n_2| k_a = p_a n_2; a^{n}_{BO3} = |n_2 b'| k_a = n_2 b'; \\ \varepsilon_3 = \dfrac{a^{\tau}_{BO3}}{BO_3} = \dfrac{|n_2 b'| k_a}{BO_3} = \dfrac{n_2 b'}{BO_3}; \varepsilon_5 = 0; \end{cases}$$

$$\begin{cases} \bar{a}_D = \bar{a}_C + \bar{a}^{n}_{DC} + \bar{a}^{\tau}_{DC} \\ \| xx' \quad \| DC \quad \perp DC \\ D \to C \end{cases} \Rightarrow \begin{cases} a_C = |p_a c'| k_a = p_a c'; a_D = |p_a d'| k_a = p_a d'; a^{n}_{DC} = |n_3 d'| k_a = n_3 d'; \\ a^{\tau}_{DC} = DC\varepsilon_4 = |c' n_3| k_a = c' n_3; \varepsilon_4 = \dfrac{a^{\tau}_{DC}}{DC} = \dfrac{|n_3 d'| k_a}{DC} = \dfrac{n_3 d'}{DC}. \end{cases}$$

5)

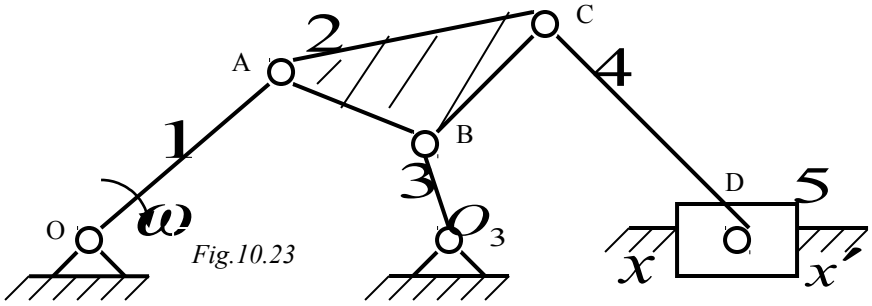

Fig.10.23

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Rezolvare:

Elementul *1* are mişcare de rotaţie, elementul *2* are mişcare plan – paralelă, elementul *3* are mişcare de rotaţie, elementul *4* are mişcare plan – paralelă, elementul *5* are mişcare de translaţie.

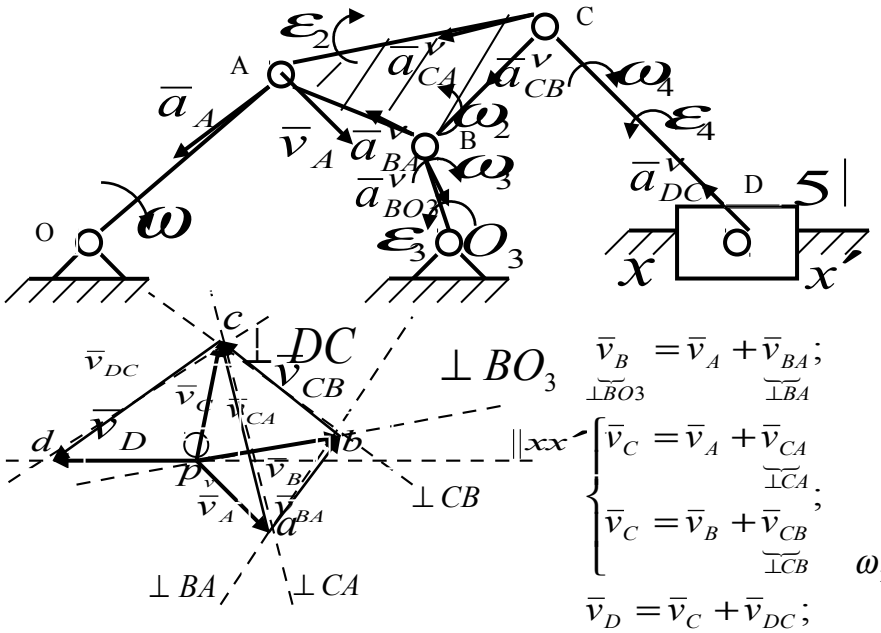

Fig.10.23

$$\bar{v}_B = \bar{v}_A + \bar{v}_{BA};\quad \perp BO_3 \quad \perp BA$$

$$\begin{cases} \bar{v}_C = \bar{v}_A + \bar{v}_{CA}; & \perp CA \\ \bar{v}_C = \bar{v}_B + \bar{v}_{CB}; & \perp CB \end{cases}$$

$$\bar{v}_D = \bar{v}_C + \bar{v}_{DC};\quad \parallel xx' \quad \perp DC$$

$$v_{BA} = \frac{(ab)k_v}{BA} = \frac{ab}{BA};$$

$$v_{CA} = \frac{(ac)k_v}{CA} = \frac{ac}{CA};$$

$$v_{CB} = \frac{(bc)k_v}{CB} = \frac{bc}{CB};$$

$$v_{DC} = \frac{(cd)k_v}{DC} = \frac{cd}{DC};$$

$$\omega_2 = \frac{v_{BA}}{BA} = \frac{ab}{BA}\frac{v_{CA}}{CA} = \frac{v_{CB}}{CB};$$

$$\omega_3 = \frac{v_B}{BO_3} = \frac{(p_v b)k_v}{BO_3} = \frac{p_v b}{BO_3};$$

$$\omega_4 = \frac{v_{DC}}{DC} = \frac{(cd)k_v}{DC} = \frac{cd}{DC}.$$

$$a_A = a^{\nu}_A = OA\omega^2 = (p_a a')k_a = p_a a';$$

$$\bar{a}_B = \bar{a}_A + \bar{a}^{\nu}_{BA} + \bar{a}^{\tau}_{BA};\quad \parallel BA \quad \perp BA$$

$$\bar{a}_B = \bar{a}_{BO_3} + \bar{a}^{\nu}_{BO_3} + \bar{a}^{\tau}_{BO_3} = \bar{a}^{\nu}_{CA} + \bar{a}^{\tau}_{CA};\quad \bar{a}_{BO_3} = 0;\quad \parallel BO_3 \quad \perp BO_3 \quad \parallel CA \quad \perp CA$$

$$\bar{a}_C = \bar{a}_B + \bar{a}^{\nu}_{CB} + \bar{a}^{\tau}_{CB};\quad \parallel CB \quad \perp CB$$

$$a_B : |p_a b'|k_a : p_a b'; a^{\nu}_{BA} : BA\omega_2^2 : |a'n_1|k_a : a'n_1;$$

$$a^{\tau}_{BA} : |n_1 b'|k_a : n_1 b'; \omega_2 : \frac{a^{\tau}_{BA}}{BA} : \frac{|a'n_1|k_a}{BA} : \frac{a'n_1}{BA};$$

$$a_B : |p_a b'|k_a : p_a b'; a^{\nu}_{BO_3} : BO_3\omega_3^2 : |p_a n_2|k_a : p_a n_2; a^{\nu}_{CA} : CA\omega_2^2 : |a'n_2|k_a : a'n_2;$$

$$a^{\tau}_{BO_3} : |n_3 b'|k_a : n_3 b'; a^{\tau}_{CA} : |n_2 b'|k_a : n_2 b'; \omega_3 : \frac{a^{\tau}_{BO_3}}{BO_3} : \frac{|n_3 b'|k_a}{BO_3} : \frac{n_3 b'}{BO_3}; \omega_2 : \frac{a^{\tau}_{CA}}{CA} : \frac{|n_2 b'|k_a}{CA} : \frac{n_2 b'}{CA};$$

$$a^{\nu}_{CB} : CB\omega_2^2 : |b'n_4|k_a : b'n_4; a^{\tau}_{CB} : |n_4 c'|k_a : n_4 c';$$

$$\omega_2 : \frac{a^{\tau}_{CB}}{CB} : \frac{|n_4 c'|k_a}{CB} : \frac{n_4 c'}{CB} : \frac{a^{\tau}_{BA}}{BA} : \frac{a^{\tau}_{CA}}{CA};$$

$$\overline{a}_D = \overline{a}_C + \overline{a}^{\nu}_{DC} + \overline{a}^{\tau}_{DC};$$
$$\overline{\|xx'} \quad \overline{\|DC} \quad \overline{\perp DC}$$
$$\overline{D\to C}$$
$$a_D = (p_a d')k_a = p_a d'$$

10.2.2. Probleme propuse

1)

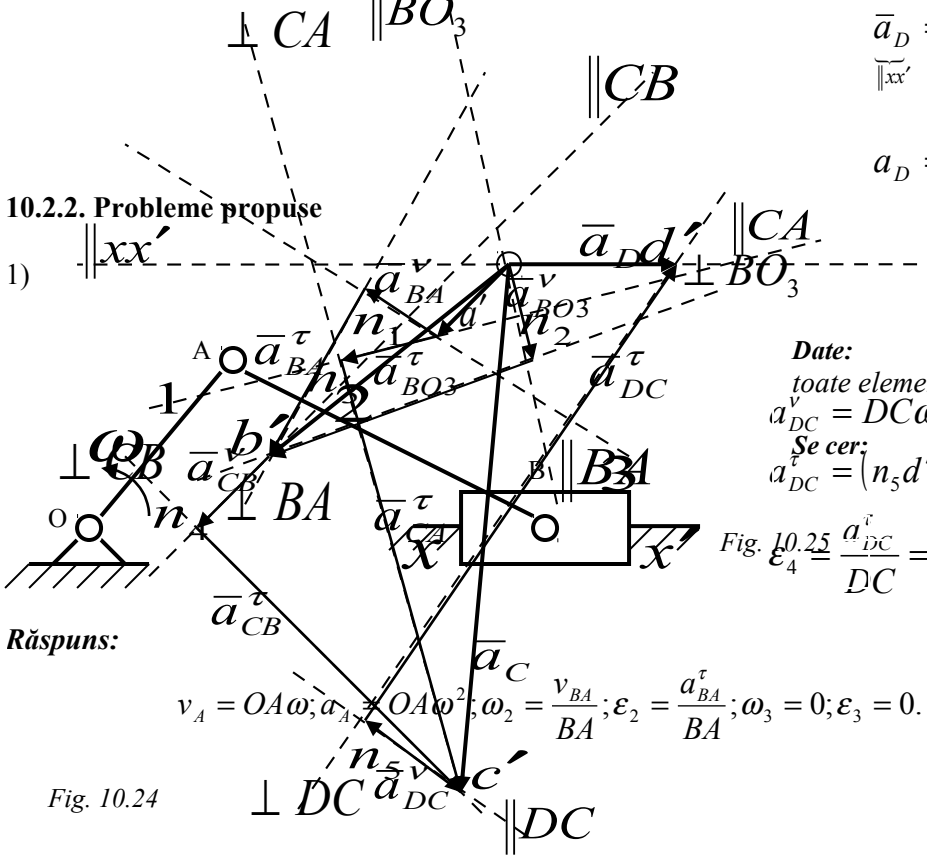

Fig. 10.24

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$a^{\nu}_{DC} = DC\omega_4^2 = (c'n_5)k_a = c'n_5;$
Se cer:
$a^{\tau}_{DC} = (n_5 d')k_a = n_5 d';$

Fig. 10.25 $\varepsilon_4 = \dfrac{a^{\tau}_{DC}}{DC} = \dfrac{(n_5 d')k_a}{DC} = \dfrac{n_5 d'}{DC}.$

Răspuns:

$$v_A = OA\omega; a_A = OA\omega^2; \omega_2 = \frac{v_{BA}}{BA}; \varepsilon_2 = \frac{a^{\tau}_{BA}}{BA}; \omega_3 = 0; \varepsilon_3 = 0.$$

2)

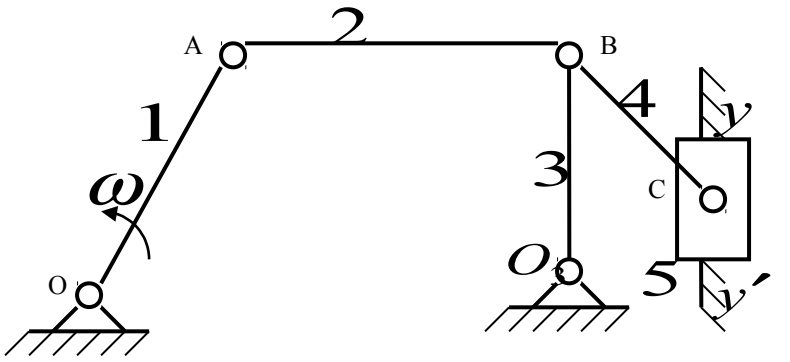

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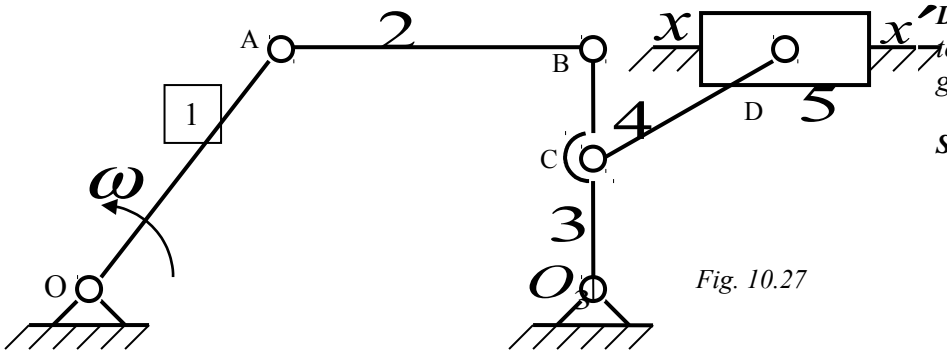
Fig. 10.26

Răspuns:

$$v_A = OA\omega; \omega_2 = \frac{v_{BA}}{BA}; \omega_3 = \frac{v_B}{BO_3}; \omega_4 = \frac{v_{CB}}{CB}; \omega_5 = 0;$$

$$a_A = OA\omega^2; \varepsilon_2 = \frac{a^{\tau}_{BA}}{BA}; \varepsilon_3 = \frac{a^{\tau}_{BO3}}{BO_3}; \varepsilon_4 = \frac{a^{\tau}_{DC}}{DC}; \varepsilon_5 = 0$$

3)



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Se cer:

Fig. 10.27

Răspuns:

$$v_A = OA\omega; \omega_2 = \frac{v_{BA}}{BA}; \omega_3 = \frac{v_B}{BO_3}; \omega_4 = \frac{v_{DC}}{DC}; \omega_5 = 0;$$

$$a_A = OA\omega^2; \varepsilon_2 = \frac{a^\tau_{BA}}{BA}; \varepsilon_3 = \frac{a^\tau_{BO3}}{BO_3}; \varepsilon_4 = \frac{a^\tau_{DC}}{DC}; \varepsilon_5 = 0.$$

4)

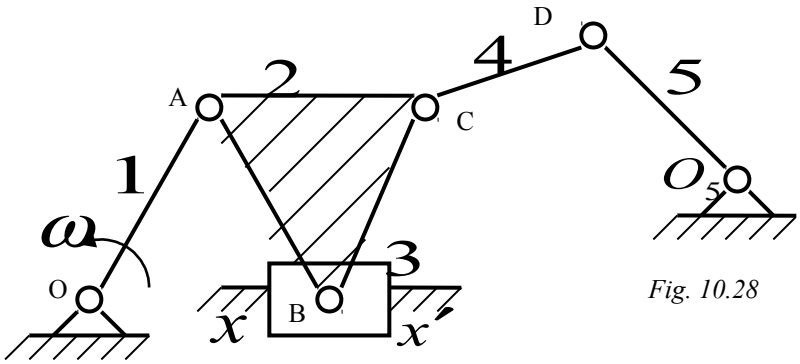

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Fig. 10.28

Răspuns:

$$v_A = OA\omega; \omega_2 = \frac{v_{BA}}{BA}; \omega_3 = 0; \omega_4 = \frac{v_{DC}}{DC}; \omega_5 = \frac{v_D}{DO_5};$$

$$a_A = OA\omega^2; \varepsilon_2 = \frac{a^\tau_{BA}}{BA}; \varepsilon_3 = 0; \varepsilon_4 = \frac{a^\tau_{DC}}{DC}; \varepsilon_5 = \frac{a^\tau_{DO5}}{DO_5}.$$

5)

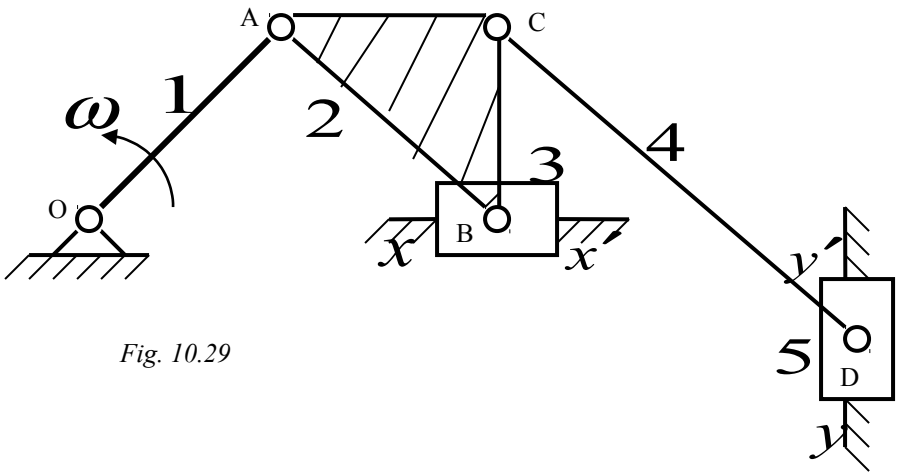

Fig. 10.29

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Răspuns:

$$v_A = OA\omega; \omega_2 = \frac{v_{BA}}{BA}; \omega_3 = 0; \omega_4 = \frac{v_{DC}}{DC}; \omega_5 = 0; a_A = OA\omega^2; \varepsilon_2 = \frac{a^\tau_{BA}}{BA}; \varepsilon_3 = 0; \varepsilon_4 = \frac{a^\tau_{DC}}{DC}; \varepsilon_5 = 0.$$

CAPITOLUL 11

CINEMATICA MIŞCĂRII RELATIVE A PUNCTULUI MATERIAL

11.1. Generalităţi. Definiţii.

1. Mişcarea absolută este mişcarea punctului material în raport cu sistemul de referinţă fix.
2. Mişcarea relativă este mişcarea punctului material în raport cu sistemul de refernință mobil ca şi cum acesta ar fi fix.
3. Mişcarea de transport este mişcarea punctului material solidarizat cu sistemul de referinţă mobil faţă de sistemul de referinţă fix.

Studiul mişcării relative a punctului material se realizează cu ajutorul următoarelor relaţii matematice:

$$\bar{v}_a = \bar{v}_r + \bar{v}_t;$$
$$\bar{a}_a = \bar{a}_r + \bar{a}_t + \bar{a}_c;$$
$$\bar{a}_C = 2\bar{\omega}_t \times \bar{v}_r.$$

11.2. Probleme rezolvate

1)

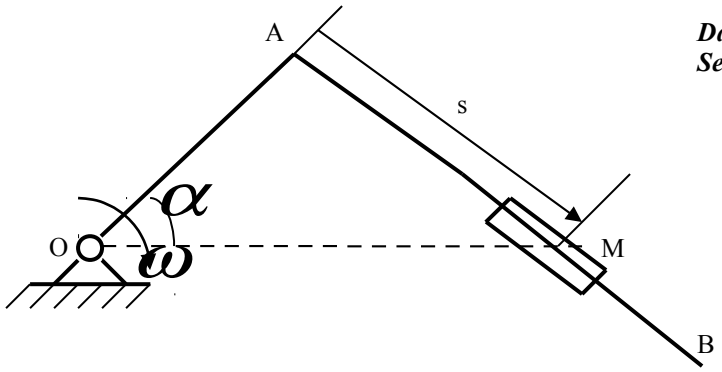

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Fig. 11.1

Rezolvare:

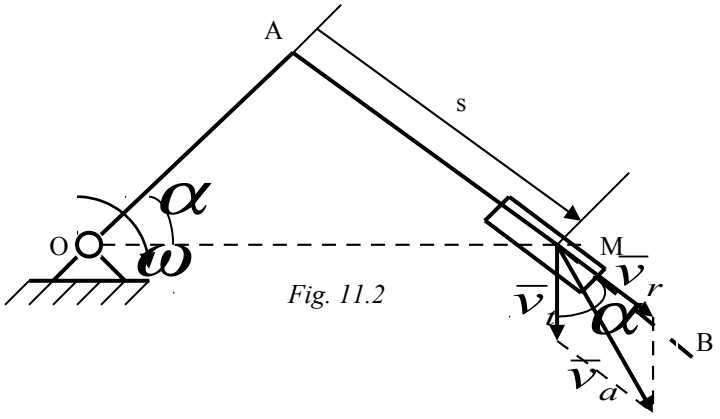

Fig. 11.2

$$\bar{v}_a = \bar{v}_r + \bar{v}_t; v_r = \dot{s} = t;$$

$$v_t = OM\omega = 2t\sqrt{l^2 + s^2};$$

$$v_a = \sqrt{v_r^2 + v_t^2 + 2v_r v_t \cos\alpha};$$

$$\cos\alpha = \frac{l}{\sqrt{l^2 + s^2}} \Rightarrow$$

$$\Rightarrow v_a = \sqrt{t^2 + 4t^2\left(l^2 + s^2\right) + 4t^2 l}.$$

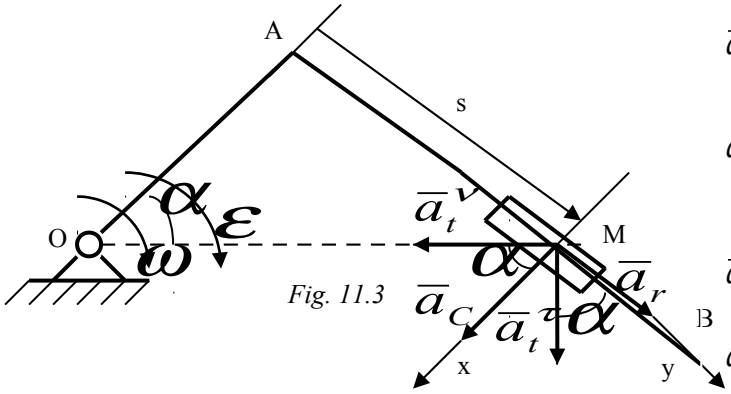

Fig. 11.3

$$\bar{a}_a = \bar{a}_r + \bar{a}_t + \bar{a}_C; a_r = \dot{v}_r = \ddot{s} = 1 m/s^2;$$

$$a_t \begin{cases} a_t^{\nu} = OM\omega^2 = 4t^2\sqrt{l^2 + s^2} \\ a_t^{\tau} = OM\varepsilon = 2\sqrt{l^2 + s^2} \end{cases};$$

$$\bar{a}_C = 2\bar{\omega} \times \bar{v}_r \Rightarrow a_C = 2\omega \cdot v_r \sin\frac{\pi}{2} = 4t^2;$$

$$a_x = a_C + a_t^{\nu}\cos\alpha + a_t^{\tau}\sin\alpha$$

$$a_y = a_r + a_t^{\tau}\cos\alpha - a_t^{\nu}\sin\alpha$$

$$\bar{a}_a = a_x\bar{i} + a_y\bar{j} \Rightarrow a_a = \sqrt{a_x^2 + a_y^2} = \sqrt{\left(a_C + a_t^{\nu}\cos\alpha + a_t^{\tau}\sin\alpha\right)^2 + \left(a_r + a_t^{\tau}\cos\alpha - a_t^{\nu}\sin\alpha\right)^2} =$$

$$= \sqrt{\left(4t^2 + 4t^2 l + 2s\right)^2 + \left(1 + 2l - 4t^2 s\right)^2} = \sqrt{\left(5t^2 + 4t^2 l\right)^2 + \left(1 + 2l - 2t^4\right)^2}$$

2)

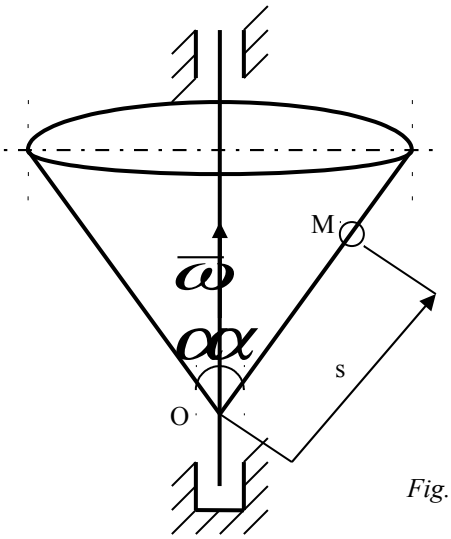

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Fig. 11.4

Rezolvare:

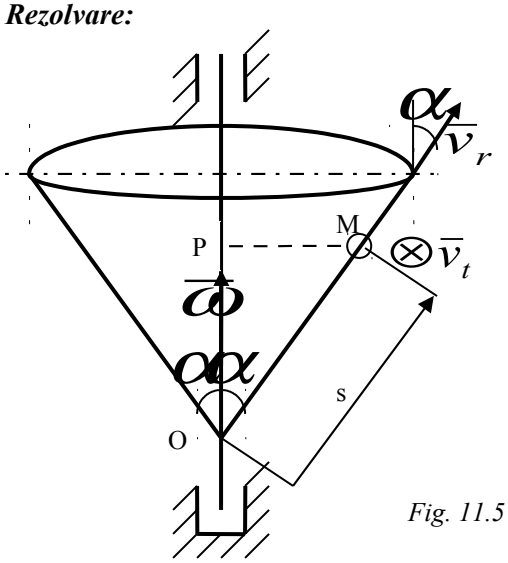

Fig. 11.5

$$\bar{v}_a = \bar{v}_r + \bar{v}_t; v_r = \dot{s} = \frac{1}{t};$$

$$v_t = MP\omega = \omega_0 s \sin\alpha = \omega_0 \ln t \sin\alpha;$$

$$v_a = \sqrt{v_r^2 + v_t^2} = \sqrt{t^{-2} + (\omega_0 \ln t \sin\alpha)^2}.$$

$$\bar{a}_a = \bar{a}_r + \bar{a}_t + \bar{a}_C; a_r = \dot{v}_r = \ddot{s} = -\frac{1}{t^2};$$

$$a_t \begin{cases} a_t^v = MP\omega^2 = \omega_0^2 \ln t \sin\alpha \\ a_t^\tau = MP\dot{\omega} = 0 \end{cases} \Rightarrow a_t = a_t^v$$

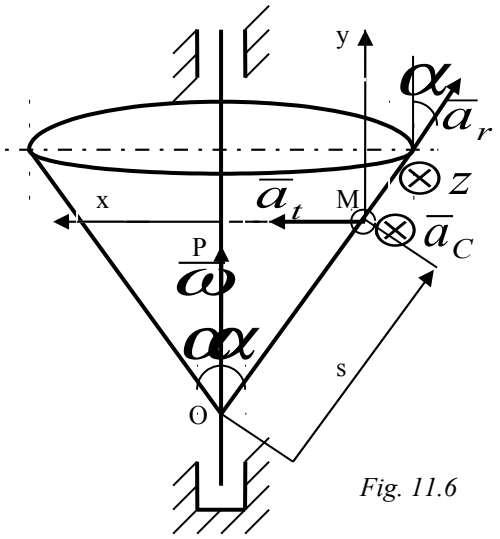

Fig. 11.6

$$\bar{a}_C = 2\bar{\omega} \times \bar{v}_r \Rightarrow a_C = 2\omega \cdot v_r \sin\alpha = 2\omega_0 \frac{1}{t} \sin\alpha;$$

$$a_x = a_t - a_r \sin\alpha \Rightarrow a_x = \omega_0^2 \ln t \sin\alpha + \frac{\sin\alpha}{t^2};$$

$$a_y = a_r \cos\alpha \Rightarrow a_y = -\frac{\cos\alpha}{t^2};$$

$$a_z = a_C \Rightarrow a_z = \frac{2\omega_0 \sin\alpha}{t}$$

$$\bar{a}_a = a_x \bar{i} + a_y \bar{j} + a_z \bar{k} \Rightarrow a_a = \sqrt{a_x^2 + a_y^2 + a_z^2} =$$

$$= \sqrt{\left(\omega_0^2 \ln t \sin\alpha + \frac{\sin\alpha}{t^2}\right) + \frac{\cos^2\alpha}{t^4} + \frac{4}{t^2}\omega_0^2 \sin^2\alpha}.$$

3)

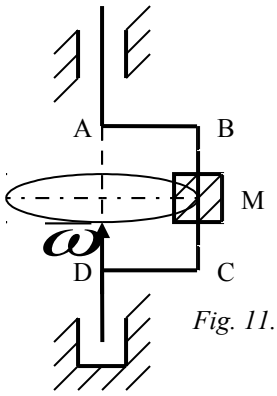

Fig. 11.7

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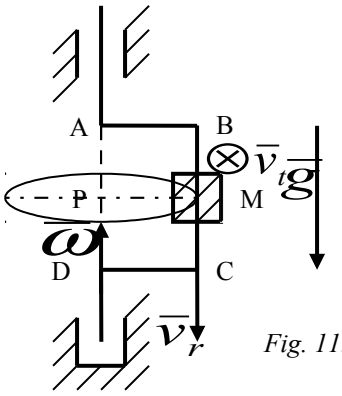

Fig. 11.8

$$\bar{v}_a = \bar{v}_r + \bar{v}_t; v_r = \int g dt = g\int dt = gt + C;$$

$$t = 0 \Rightarrow v_r = 0 \Rightarrow C = 0 \Rightarrow v_r = gt;$$

$$v_t = MP\omega = l\omega_0;$$

$$v_a = \sqrt{v_t^2 + v_r^2} = \sqrt{l^2\omega_0^2 + g^2t^2}.$$

$$\bar{a}_a = \bar{a}_r + \bar{a}_t + \bar{a}_C; a_r = \dot{v}_r = g;$$

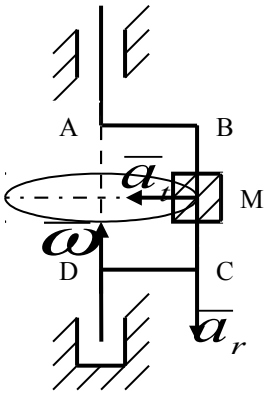

Fig. 11.9

$$a_t \begin{cases} a_t^v = MP\omega^2 = l\omega_0^2 \\ a_t^\tau = MP\varepsilon = MP\dot{\omega} = 0 \end{cases};$$

$$\bar{a}_C = 2\bar{\omega} \times \bar{v}_r \Rightarrow a_C = 2\omega_0 gt \sin\pi = 0;$$

$$a_a = \sqrt{a_r^2 + a_t^2} = \sqrt{g^2 + l^2\omega_0^4}.$$

$$\bar{a}_t = \bar{a}_t^v$$

4)

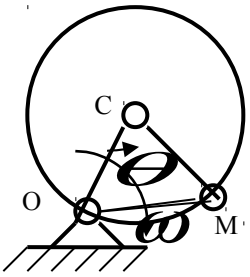

Fig. 11.10

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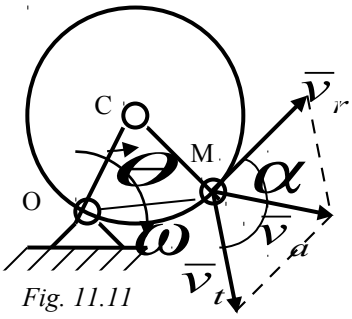

Fig. 11.11

$$\bar{v}_a = \bar{v}_r + \bar{v}_t; v_r = R\dot{\theta} = (2t-2)R; v_t = OM\omega = (2t-3)\cdot 2R\sin\frac{\theta}{2};$$

$$v_a = v_r^2 + v_t^2 + 2\cdot v_r v_t \cos\alpha; \alpha = \pi - \measuredangle(OMC) = \pi - \frac{\pi}{2} + \frac{\theta}{2} = \frac{\pi}{2} + \frac{\theta}{2};$$

$$v_a = \sqrt{v_r^2 + v_t^2 - 2\cdot v_r v_t \sin\frac{\theta}{2}};$$

$$v_a = \sqrt{4R^2(t-1)^2 + (2t-3)^2 4R^2\sin^2\frac{\theta}{2} - 8R^2(t-1)\sin^2\frac{\theta}{2}}.$$

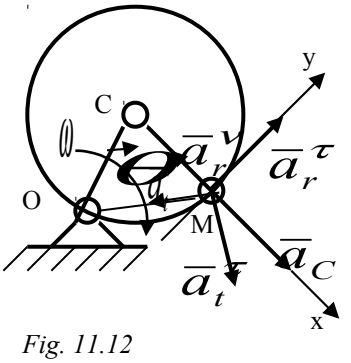

Fig. 11.12

$$\bar{a}_a = \bar{a}_r + \bar{a}_t + \bar{a}_C; a_r \begin{cases} a_r^{\nu} = R\dot{\theta}^2 = R\cdot(2t-2)^2 \\ a_r^{\tau} = R\ddot{\theta} = 2R \end{cases};$$

$$a_t \begin{cases} a_t^{\nu} = OM\omega^2 = (2t-3)^2 2R\sin\frac{\theta}{2} \\ a_t^{\tau} = OM\dot{\omega} = 2\cdot 2R\sin\frac{\theta}{2} \end{cases};$$

$$\bar{a}_C = 2\bar{\omega}\times\bar{v}_r \Rightarrow a_C = 2\omega\cdot v_r \sin\frac{\pi}{2} = 4R\cdot(t-1)\cdot(2t-3);$$

$$\bar{a}_a = a_x\bar{i} + a_y\bar{j} \Leftrightarrow a_a = \sqrt{a_x^2 + a_y^2}$$

$$a_x = a_C + a_t^{\tau}\cos\frac{\theta}{2} - a_t^{\nu}\sin\frac{\theta}{2} - a_r^{\nu}$$

$$a_y = a_r^{\tau} - a_t^{\tau}\sin\frac{\theta}{2} - a_t^{\nu}\cos\frac{\theta}{2}$$

5)

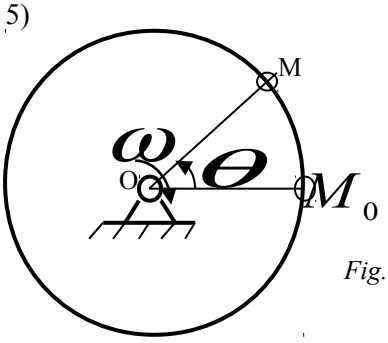

Fig. 11.13

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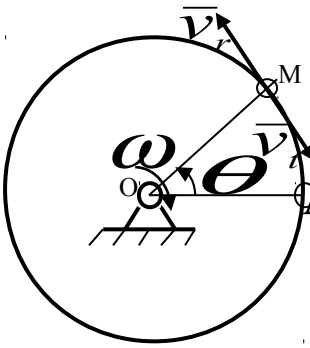

Fig. 11.14

$$\bar{v}_a = \bar{v}_r + \bar{v}_t; v_a \begin{cases} v_r = R\dot{\theta} = R\varepsilon_0 t \\ v_t = R\omega = R\varepsilon_0 t \end{cases} \Rightarrow v_a = v_r - v_t = 0$$

$$\bar{a}_a = \bar{a}_r + \bar{a}_t + \bar{a}_C;$$

$$a_r \begin{cases} a_r^{\nu} = R\dot{\theta}^2 = R\varepsilon_0^2 t^2 \\ a_r^{\tau} = R\ddot{\theta} = R\varepsilon_0 \end{cases}; a_t \begin{cases} a_t^{\nu} = R\omega^2 = R\varepsilon_0^2 t^2 \\ a_t^{\tau} = R\dot{\omega} = R\varepsilon_0 \end{cases}$$

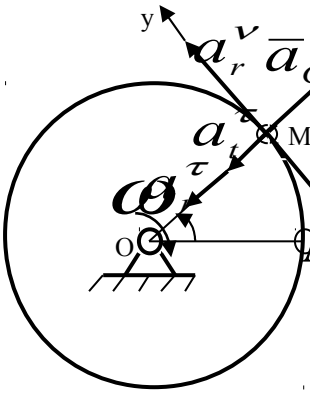

Fig. 11.15

$$\bar{a}_C = 2\bar{\omega} \times \bar{v}_r \Rightarrow a_C = 2\omega \cdot v_r \sin\frac{\pi}{2} = 2R\varepsilon_0^2 t^2;$$

$$a_x = a_C - a_t^{\tau} - a_r^{\tau} = 0$$

$$a_y = a_r^{\nu} - a_t^{\nu} = 0$$

$$\bar{a} = a_x \bar{i} + a_y \bar{j} = 0 \Leftrightarrow a = \sqrt{a_x^2 + a_y^2} = 0$$

6)

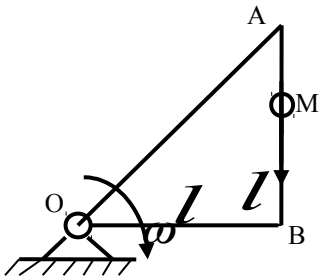

Fig. 11.16

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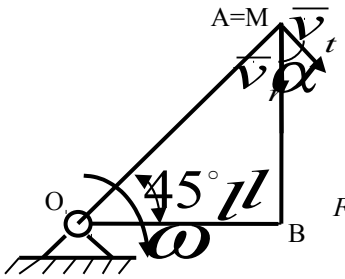


Fig. 11.17

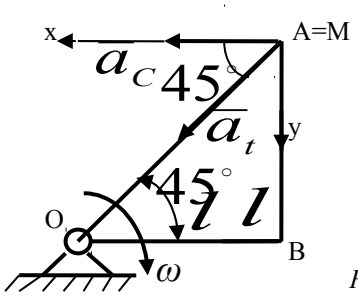
$$\bar{v}_a = \bar{v}_r + \bar{v}_t;$$

$$2\pi = \omega \cdot t \Rightarrow t = \frac{2\pi}{\omega}; l = v_r t \Rightarrow v_r = \frac{l}{t} = \frac{l\omega}{2\pi};$$

$$v_t = OA\omega = l\omega\sqrt{2};$$

$$v_a = \sqrt{v_r^2 + v_t^2 + 2v_r v_t \cos 45^\circ}$$

$$v_a = \frac{l\omega}{2\pi}\sqrt{8\pi^2 + 4\pi + 1}$$



$$\bar{a}_a = \bar{a}_r + \bar{a}_t + \bar{a}_C; a_r = \dot{v}_r = 0; a_t \begin{cases} a_t^v = OA\omega^2 = l\sqrt{2}\omega^2 \\ a_t^\tau = OA\dot{\omega} = 0 \end{cases}$$

$$\bar{a}_t = a_t^v; \bar{a}_C = 2\bar{\omega} \times \bar{v}_r \Rightarrow a_C = 2\omega \cdot v_r \sin\frac{\pi}{2} = 2\omega\frac{l\omega}{2\pi} = \frac{l}{\pi}\omega^2$$

Fig. 11.18

$$a_a = \sqrt{a_t^2 + a_C^2 + 2a_t a_C \cos 45^\circ} = \frac{l\omega^2}{\pi}\sqrt{2\pi^2 + 2\pi + 1};$$

$$a_x = a_C + a_t \cos 45^\circ = \frac{l}{\pi}\omega^2 + l\omega^2;$$

$$a_y = a_t \sin 45^\circ = l\omega^2;$$

$$\bar{a}_a = a_x\bar{i} + a_y\bar{j} = \left(\frac{l}{\pi}\omega^2 + l\omega^2\right)\cdot\bar{i} + l\omega^2 \cdot \bar{j}.$$

11.3. Probleme propuse

1)

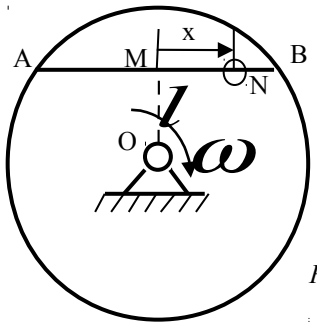


Fig. 11.19

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Răspuns:

2)

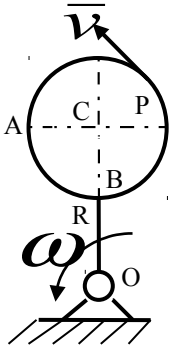

Fig. 11.20

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3)

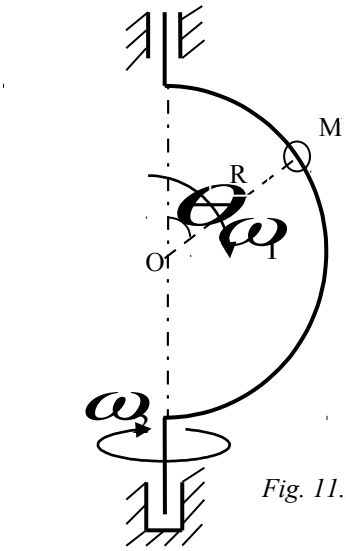

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Fig. 11.21

4)

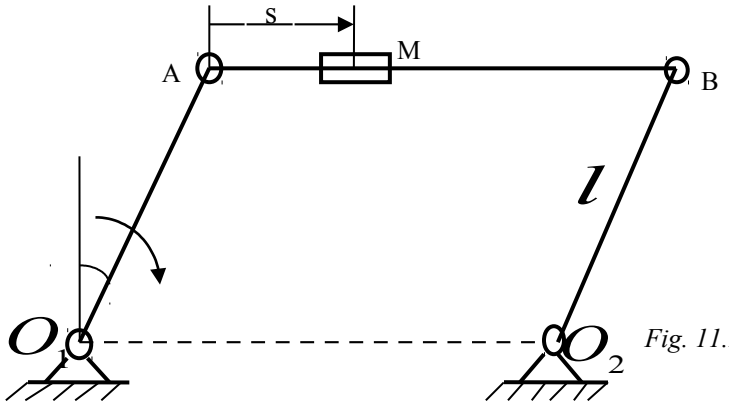

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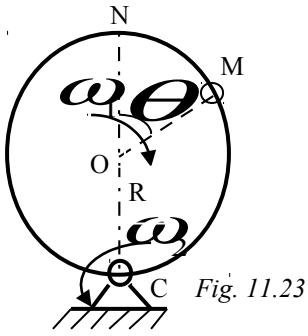
Fig. 11.22

Răspuns:

$$v_a = \sqrt{36t^2 + \omega^2 t^2 + 12tl\omega\cos\theta}; a_a = \sqrt{36 + l^2\omega^4 - 12l\omega^2\sin\theta}.$$

5)



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Răspuns:

Fig. 11.23

CAPITOLUL 12

CINEMATICA MIŞCĂRII RELATIVE A SOLIDULUI RIGID

12.1. Probleme rezolvate

1)

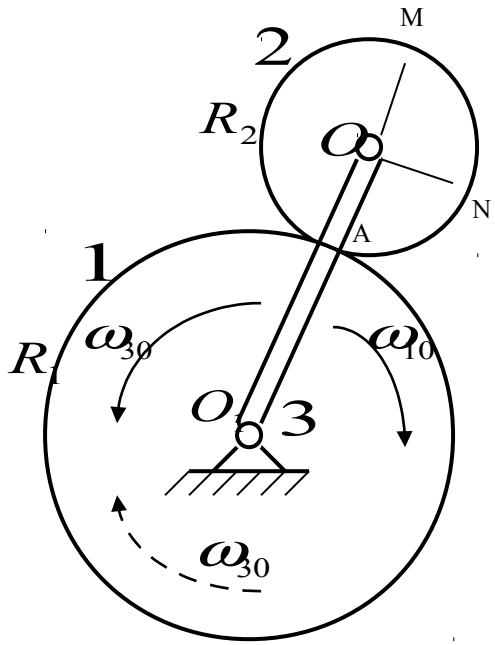

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Fig. 12.1

Rezolvare:

Mişcările corpurilor sunt rotații în plan *1*,*2* şi plan – paralelă *2*, vitezele unghiulare fiind paralele între ele, adică perpendiculare pe plan, rezultă că mişcarea compusă a corpului *2* este o sumă de rotații paralele. Considerând ca sens pozitiv, sensul orar al lui ω_{10}, valoarea vitezei unghiulare absolute a corpului *2* se determină cu una dintre relațiile:

$$\begin{aligned} \omega_{20} &= \omega_{21} + \omega_{10}; \\ \omega_{20} &= \omega_{23} - \omega_{30}. \end{aligned} \tag{12.1}$$

Dar în cazul rotațiilor paralele rezultă pentru corpul *2* o rotație instantanee, în jurul centrului vectorilor paraleli $\overline{\omega}$. Notând acest punct cu *C*, se poate aplica vectorilor teorema lui Varignon în raport cu punctul *O*. Din relația precedentă se observă că teorema lui Varignon poate fi aplicată de două ori pentru fiecare sumă de vectori paraleli:

$$\begin{aligned} \omega_{20} \cdot OC &= \omega_{21} \cdot O_1A + \omega_{10} \cdot 0; \\ \omega_{20} \cdot OC &= \omega_{23} \cdot O_1O - \omega_{30} \cdot 0. \end{aligned} \tag{12.2}$$

În membrul drept al relațiilor de mai sus apar distanțele față de O_1 ale puntelor din plan aflate pe suporturile celor patru vectori: $\overline{\omega}_{10}$ şi $\overline{\omega}_{30}$ trec prin O_1; $\overline{\omega}_{21}$ trece prin *A* (A reprezintă punctul de viteză nulă între roțile 2 şi 1, în acest punct neexistând alunecare între cele două roți); ω_{23} trece prin ponctul O, acesta fiind punctul de viteză nulă între corpurile 2 şi 3 (O este articulația comună). Deci, relațiile de mai sus devin:

$$\left.\begin{aligned}\omega_{20}\cdot OC &= 2R\omega_{21}\\ \omega_{20}\cdot OC &= 3R\omega_{23}\end{aligned}\right\}\Rightarrow 2\omega_{21}=3\omega_{23}$$

Înlocuind acest rezultat în relațiile (12.1), se obține:

$$\left.\begin{aligned}\omega_{21}+\omega_{10} &= \omega_{23}-\omega_{30}\\ \omega_{21} &= \frac{3}{2}\omega_{23}\end{aligned}\right\} \tag{12.3}$$

Din sistemul format din ecuațiile (12.4) şi (12.1) rezultă:

$$\omega_{21}=-5\omega_0;\ \omega_{23}=-\frac{10}{3}\omega_0;\ \omega_{20}=-4\omega_0;\ O_1C=\frac{5}{2}R$$

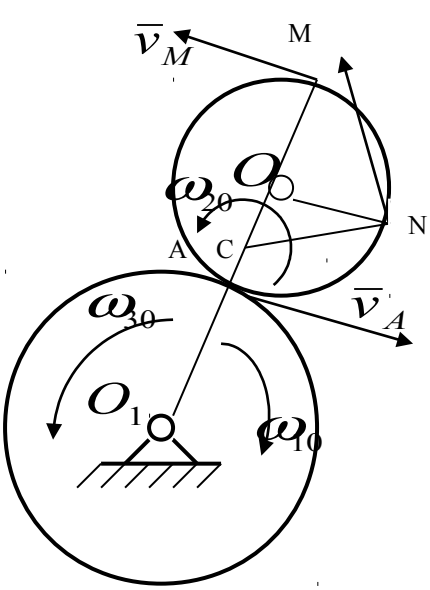

Fig. 12.2

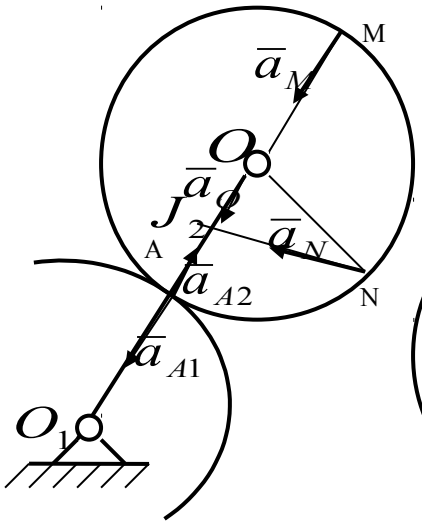

Fig. 12.3

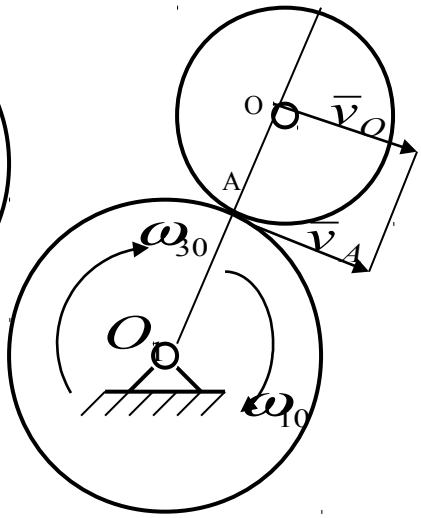

Fig. 12.4

În figura 12.2 se arată distribuția de viteze a roții *2*, față de punctul *C*, care devine pentru această roată, centru instantaneu de rotație față de sistemul de referință fix. Vitezele reprezentate pe figură pentru roata *2* sunt de forma:

$$v_M=\omega_{20}\cdot CM=6R\omega_0;\ v_N=\omega_{20}\cdot CN=2\sqrt{5}R\omega_0;\ v_A=\omega_{20}\cdot CA=2R\omega_0.$$

Calculând viteza punctului *A* al roții *1* $\left(v_A=\omega_{10}\cdot O_1A\right)$, rezultă un vector cu aceeaşi direcție, sens şi modul cu cea calculată pentru punctul *A* al roții *2*, deci se verifică faptul că punctul *A* este centru instantaneu al rotației relative între roțile *1* şi *2*.

Pentru determinarea accelerațiilor punctelor *M* şi *N*, se scrie:

$$\left.\begin{aligned}\bar{a}_M &= \bar{a}_O+\bar{\varepsilon}_{20}\times\overline{OM}-\omega_{20}^2\cdot\overline{OM};\\ \bar{a}_N &= \bar{a}_O+\bar{\varepsilon}_{20}\times\overline{ON}-\omega_{20}^2\cdot\overline{ON}.\end{aligned}\right\} \tag{12.4}$$

Accelerația unghiulară absolută a roții *2* este:

$$\bar{\varepsilon}_{20} = \frac{\partial \bar{\omega}_{10}}{\partial t} + \frac{\partial \bar{\omega}_{21}}{\partial t} + \bar{\omega}_{10} \times \bar{\omega}_{21} = \frac{\partial \bar{\omega}_{30}}{\partial t} + \frac{\partial \bar{\omega}_{23}}{\partial t} + \bar{\omega}_{30} \times \bar{\omega}_{23} = 0$$

S-a scris expresia vectorială a acceleraţiei unghiulare absolute a roţii *2* pornind de la relaţiile de mişcare ale celor două corpuri *1* şi *3*, cu care corpul *2* este în legătură. Acceleraţia unghiulară este nulă, deoarece vectorii viteză unghiulară de la care se porneşte au direcţii şi mărimi care nu se modifică în timpul mişcării. Relaţiile (12.4) devin:

$$\begin{aligned} \bar{a}_M &= \bar{a}_O - \omega_{20}^2 \cdot \bar{O}\bar{M}; \\ \bar{a}_N &= \bar{a}_O - \omega_{20}^2 \cdot \bar{O}\bar{N}. \end{aligned} \tag{12.5}$$

Se determină:

$$a_O = O_1O \cdot \omega_{30}^2 = \frac{4}{3} R\omega_0^2;$$

$$a_M = a_O + OM \cdot \omega_{20}^2 = \frac{52}{3} R\omega_0^2;$$

$$a_N = \sqrt{a_O^2 + \left(\omega_{20}^2 \cdot ON\right)^2} = \frac{\sqrt{2320}}{3} R\omega_0^2.$$

Acceleraţiile determinate ale punctelor *O, M, N, A* ($\bar{a}_{A2}$, punctul *A* aparţinând roţii *2*) sunt reprezentate în figura 12.3.

Mişcarea roţii *2* cu centrul în *O* fiind plan – paralelă, cu acceleraţie unghiulară nulă, rezultă că toate acceleraţiile roţii 2 vor converge spre centrul instantaneu al acceleraţiilor (CIA) J_2, a cărui poziţie faţă de punctul *O* este definită de relaţia:

$$OJ_2 = OJ = \frac{a_O^v}{\omega_{20}^2} = \frac{R}{12}.$$

Acceleraţia a_{A2} se calculează în două moduri astfel (figura 12.3):

$$a_{A2} = \omega_{20}^2 \cdot OA - a_O = \omega_{20}^2 \cdot J_2A = \frac{44}{3} R\omega_0^2.$$

Acceleraţia a_{A1}, pentru roata 1, este

$$a_{A1} = O_1A \cdot \omega_{10}^2 = 2R\omega_0^2,$$

îndreptată spre centrul O_1, deci $\bar{a}_{A1} \neq \bar{a}_{A2}$, doar vitezele fiind egale.

Pentru inversarea sensului lui ω_{30} :

Din relaţiile (12.2) rezultă: $\omega_{21} = -\omega_0; \omega_{23} = -\frac{2}{3}\omega_0$, pe care înlocuindu-le în relaţiile (12.1) se obţine $\omega_{20} = 0$, deci mişcarea absolută a roţii *2* este de translaţie. Vitezele, respectiv acceleraţiile tuturor punctelor roţii *2* sunt egale, ele putând fi calculate pentru punctul *O*:

$$v_O = v_2 = O_1O \cdot \omega_{30} = 2R\omega_0;$$

$$a_O = a_2 = O_1O \cdot \omega_{30}^2 = \frac{4}{3} R\omega_0^2.$$

Observație:

Din calculul vitezelor în punctele *A* şi *O* se observă că roata *2* are o mişcare de translație (figura 12.4) adică:

$$v_A = O_1 A \cdot \omega_{10} = 2R\omega_0;$$
$$v_O = O_1 O \cdot \omega_{30} = 2R\omega_0.$$

2)

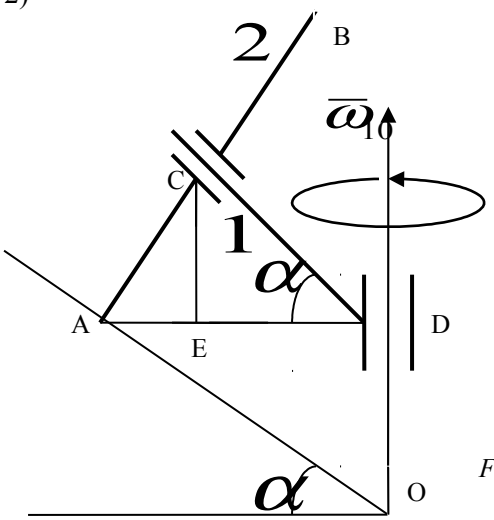

Fig. 12.5

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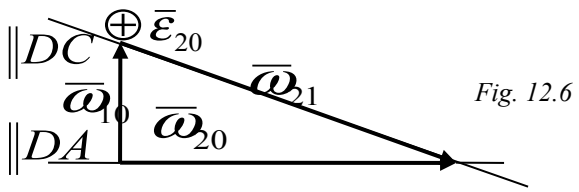

Fig. 12.6

Rezolvare:

Viteza unghiulară $\overline{\omega}_{21}$, din rotația discului *2* față de bara *1*, are drept suport axa *CD*. Deoarece discul 2 se roteşte în raport cu acesta, pe axa *CD* se găsesc toate punctele de viteză nulă ale discului față de axă; pentru axa *CD*, punctele de viteză nulă se găsesc pe axa verticală *OD*. În aceste condiții, punctul *D* este un punct de viteză nulă al discului în raport cu sistemul de referință fix. Al doilea punct de viteză nulă al discului față de sistemul de referință fix este *A*, deoarece discul se rostogoleşte fără să alunece pe suprafața conică. Rezultă de aici că dreapta *AD* reprezintă suportul vitezei unghiulare $\overline{\omega}_{20}$ a discului *2* față de sistemul de referință fix, fiind definită la un moment dat de două puncte de viteză absolută nulă ale discului. Pe baza analizei anterioare se poate scrie (fig. 12.6):

$$\overline{\omega}_{20} = \overline{\omega}_{10} + \overline{\omega}_{21}$$

Pe baza reprezentării grafice, rezultă:

$$\omega_{20} = \omega_{10} ctg\alpha = \omega_0 \sqrt{3}; \omega_{21} = \frac{\omega_{10}}{\sin\alpha} = 2\omega_0.$$

Pentru calculul vitezei lui A, se ține seama de rotația instantanee a discului față de dreapta AD, deci:

$$v_A = v_B = \omega_{20} \cdot CE = 3R\omega_0 \Leftrightarrow CE = 2R\cos\alpha = R\sqrt{3}$$

Expresia vectorială a acceleraţiei unghiulare absolute a discului 2 este de forma:

$$\bar{\varepsilon}_{20} = \frac{d\bar{\omega}_{20}}{dt} = \frac{d\bar{\omega}_{10}}{dt} + \frac{\partial\bar{\omega}_{21}}{\partial t} + \bar{\omega}_{21} \times \bar{\omega}_{10};$$

unde:

$$\frac{d\bar{\omega}_{10}}{dt} = 0 \Leftrightarrow \bar{\omega}_{10} = ct;$$

$\frac{\partial\bar{\omega}_{21}}{\partial t} = 0,$ deoarece fiind o derivată locală, se referă la rotaţia discului în raport cu bara *CD*, care pentru disc reprezintă un reper fix, deci în raport cu acest reper

Aşadar:

$$\bar{\varepsilon}_{20} = \bar{\omega}_{21} \times \bar{\omega}_{10}; \varepsilon_{20} = \omega_{21} \cdot \omega_{10} \cdot \sin \measuredangle(\bar{\omega}_{21}, \bar{\omega}_{10}) \Rightarrow \varepsilon_{20} = \omega_{21}\omega_{10} \sin\frac{\pi}{2} = \omega_0^2\sqrt{3}.$$

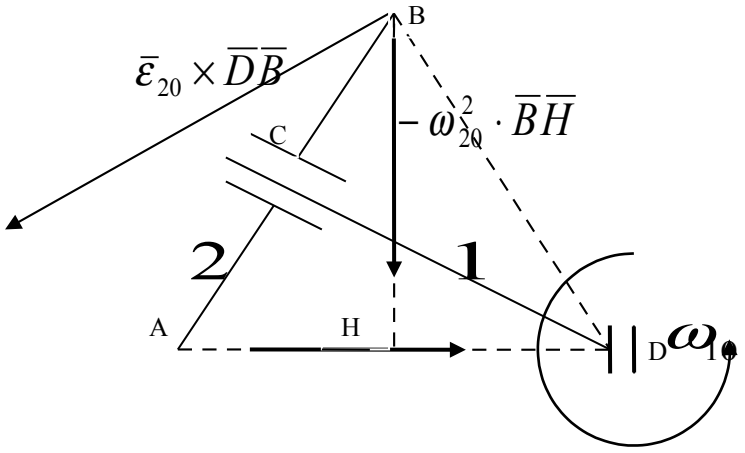

Fig. 12.7

Expresia vectorială a acceleraţiei lui *B* faţă de punctul fix *D* de pe axa instantanee de rotaţie este:

$$\bar{a}_B = \bar{\varepsilon}_{20} \times \overline{DB} - \omega_{20}^2 \cdot \overline{HB}$$

unde:

$$a_1 = \left|\bar{\varepsilon}_{20} \times \overline{DB}\right| = \varepsilon_{20} \cdot DB \cdot \sin\frac{2\pi}{3} = 3R\omega_0^2$$

$$a_2 = \left|-\omega_{20}^2 \overline{HB}\right| = 3\sqrt{3}R\omega_0^2$$

Rezultă:

$$a_B = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos\frac{\pi}{3}} = 3R\omega_0^2\sqrt{4+\sqrt{3}}$$

12.2. Probleme propuse

1)

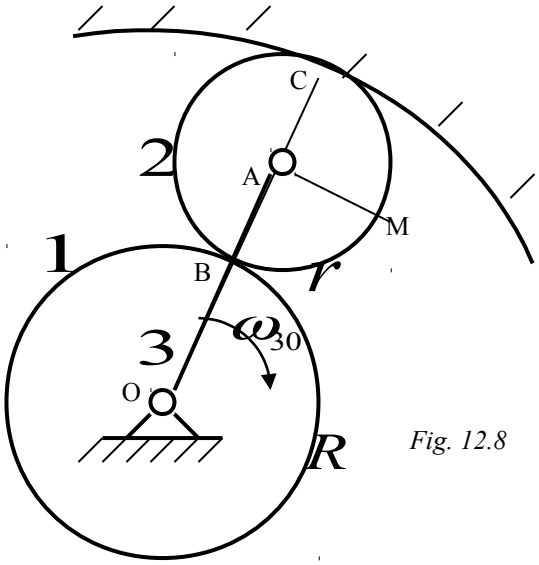

Fig. 12.8

Date:
Rostogolire fără alunecare;R; r;

Se cer:

Indicație:

Față de problema 12.1, la care nu se cunoştea punctul prin care suportul lui $\overline{\omega}_{2o}$ intersectează planul figurii, la această problemă el este cunoscut : punctul de tangență între roata *2* şi suprafața cilindrică fixă, acesta fiind centrul instantaneu de rotație absolut al roții *2*.

Răspuns:

$$\omega_{10}=2\varepsilon_0 t\frac{R+r}{R};\varepsilon_{10}=2\varepsilon_0\frac{R+r}{R};\omega_{20}=-\varepsilon_0 t\frac{R+r}{r};\varepsilon_{20}=-\varepsilon_0\frac{R+r}{r};v_M=\varepsilon_0 t\sqrt{2}(R+r)\perp CM;$$

$$a_M=\varepsilon_0\frac{R+r}{r}\sqrt{\varepsilon_0^2 t^4\left[r^2+(R+r)^2\right]+r^2\left(1-2\varepsilon_o t^2\right)}$$

2)

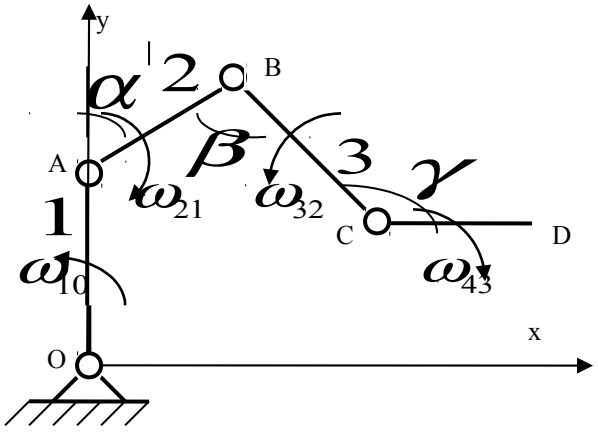

Fig. 12.9

Date:

Se cer:

Răspuns:

$$\bar{v}_D = \frac{l\omega_0}{2}\left[\left(\sqrt{3}-1\right)\cdot\bar{i}+\left(2-\sqrt{3}\right)\cdot\bar{j}\right]; \bar{a}_D = -\frac{l\omega_0^2}{4}\left[\left(3+2\sqrt{3}\right)\cdot\bar{i}+\left(6-2\sqrt{3}\right)\cdot\bar{j}\right].$$

3)

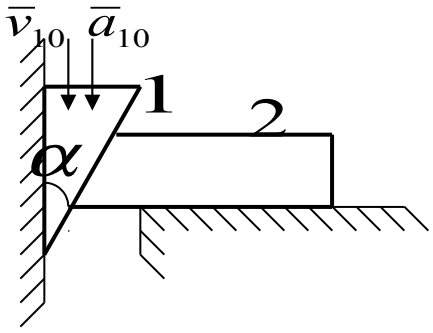

Fig. 12.10

Date:

Se cer:

Răspuns:

$$v_{20} = v_{10}\cdot tg\alpha; a_{20} = a_{10}\cdot tg\alpha;$$

Ambele sunt orientate spre dreapta.

4)

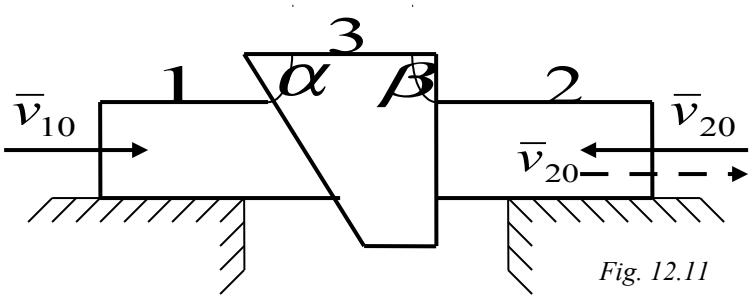

Fig. 12.11

Date:

Se cer:
, pentru ambele sensuri ale lui

Răspuns:

$$v_{30} = \sqrt{u_1^2 + v_{20}^2 - 2u_1 v_{20}\cos\beta}; tg\delta = -\frac{u_1\sin\beta}{v_{20}-u_1\cos\beta}; u_1 = \frac{v_{10}+v_{20}}{\sin(\alpha+\beta)}\cdot\sin\alpha;$$

$$v_{30} = \sqrt{u_2^2 + v_{10}^2 + 2u_2 v_{10}\cos\alpha}; tg\delta = -\frac{u_2\sin\alpha}{v_{10}+u_2\cos\alpha}; u_2 = \frac{v_{20}-v_{10}}{\sin(\alpha+\beta)}\cdot\sin\beta.$$

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